S.Y.B.Sc. : Semester - III

US03CMTH21

Numerical Methods

[Syllabus effective from June , 2019]

Study Material Prepared by :
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Unit:1
Errors and Their Computations,A General Error Formula,Errors in a series approximation, Solutions of Algebraic and Transcendental Equations : Bisection Method , Iteration Method, Aitken's Δ^2 Process , Method of False Position , Newton-Raphson Method , Ramanujan's Method.

Unit:2
Interpolation : Finite Differences, Forward ,Backward and Central Differences , Symbolic Relations of Operators , Detection of Errors by Use of Difference Tables , Differences of a Polynomial , Newton's Forward and Backward Formulae , Gauss Forward and Backward Formulae , Stirling's ,Bessel's and Everett's Formulae.

Unit:3
Interpolation with Unequally Spaced Points , Lagrange's Interpolation Formula (Without proof) , Divided Difference and Their Properties , Newton's General Interpolation Formula , Interpolation by Iteration , Inverse Interpolation , Method of Successive Approximations , Numerical Differentiation:- Newton's Forward and Backward, Gauss's Method , Maximum and Minimum Values of a Tabulated Function.

Unit:4
Numerical Integration :- Trapezoidal Rule , Simpson's $\left(\frac{1}{3}\right)^{rd}$ and $\left(\frac{3}{8}\right)^{th}$ Rules , Romberg Integration , Numerical Solution of Ordinary Differential Equation by Taylor's Series, Picards' Method , Euler's Method , Modified Euler's Method , Range-Kutta Method.

Recommended Textbooks :

1. Introductory Methods of Numerical Analysis
   Author : S.S.Sastry
   Edition : 1990
   Publisher : Prentice Hall of India

Recommended Reference Books :

1. Numerical Analysis
   Author : G.Sankar Rao
   Edition : 1997
   Publisher : New Age International (P) Liited, Publishers, New Delhi

2. Numerical Analysis
   Author : B.S.Garewal
   Edition :
   Publisher :

US03CMTH21- UNIT : I

1. Discuss the Bisection method for approximation of root of an equation.

Bisection Method

Let $f(x) = 0$ be an equation such that the function $f(x)$ is continuous on an interval $[a, b]$ and $f(a)$ and $f(b)$ have opposite signs.
Therefore there is a real root, say ξ, of the equation in the interval (a, b) (i.e. between a and b).

First let us find the mid-value, sayx_1, between a and b.

$$x_1 = \frac{a+b}{2}$$

We can treat x_1 as an approximation of actual real root ξ as it is the mid-value of the interval.

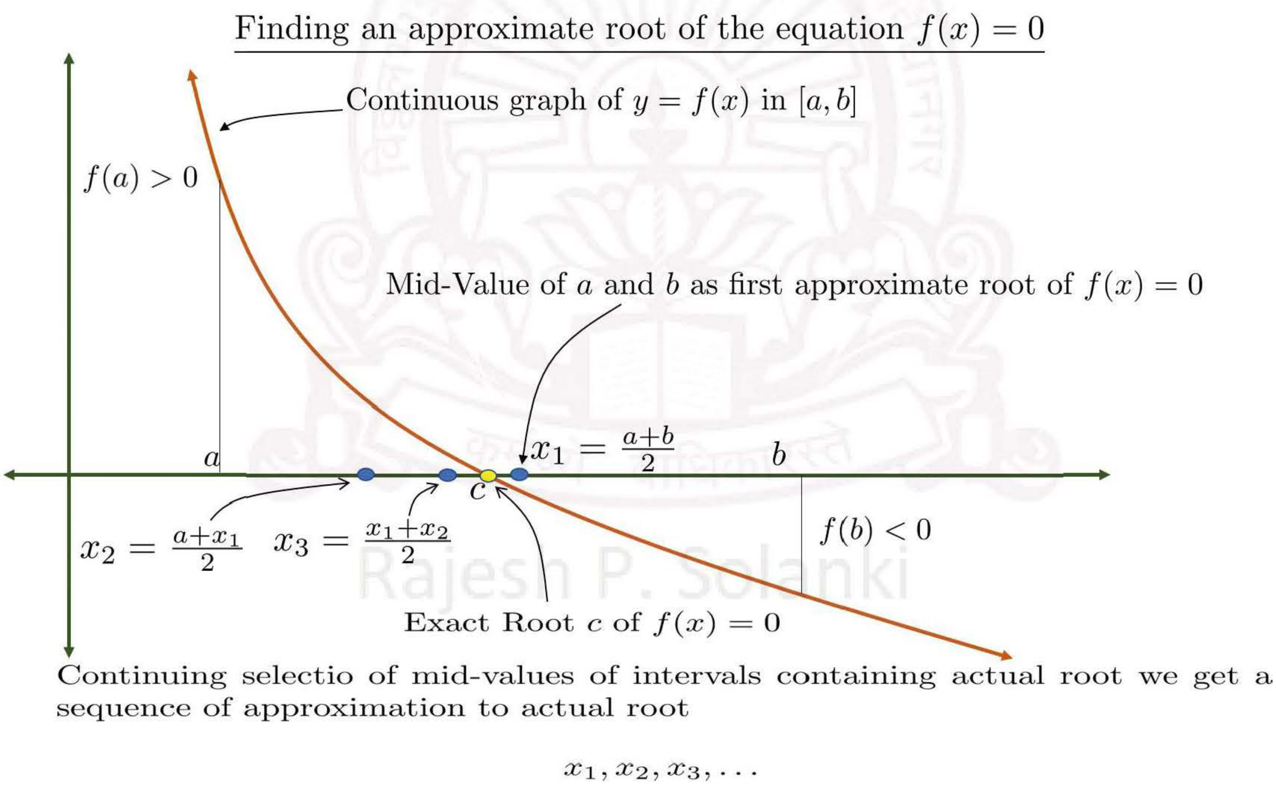


FIGURE 1. Approximating a root using Bisection method

Now, if $f(x_1)$ and $f(a)$ (or $f(b)$) have the same sign then $f(x_1)$ and $f(b)$ (or $f(a)$) are of opposite signs. Therefore the real root ξ must be in $[x_1, b]$ (or $[a, x_1]$) whose length is $\frac{|b-a|}{2}$ and is half of the length of the original interval.

Thus, the original interval is bisected and one of the sub-interval contains the actual root ξ.

So, the next approximation x_2 of the real root can be ontained as mid-value of $[x_1, b]$ (or $[a, x_1]$)

Therefore, $x_2 = \dfrac{x_1 + b}{2}$ or $x_2 = \dfrac{a + x_1}{2}$, depending on the interval.

Again we evaluate $f(x_2)$ and ,as discussed above, depending on its sign we replace one of the end-points of the current interval and obtain new interval of approximation whose mid-value x_3 will be next apprximate value of the actual root ξ.

Continuing similarly we get a sequance of approximations $x_1, x_2, x_3, \ldots$ shch that the approximations get closer and closed to ξ (because successively subinterval lengths are becoming smaller and smaller.)

If we want to approximate ξ correct upto m decimal places then we shall stop at a stage where for some positive integer n we find that the first m digits immediately after the decimal points in x_n adn x_{n+1} are identical. In that case we can say that x_n is correct upto m decimal places.

2. **Using Bisection method find a real root of the equation $x^3 + x^2 - 1 = 0$ correct upto three decimal palaces**

Solution:
For $f(x) = x^3 + x^2 - 1$, we have $f(0) = -1$ and $f(1) = 1$.
Therefore, a root of the equation $x^3 + x^2 - 1 = 0$ lies in $(0, 1)$

Let $X_1 = \dfrac{0+1}{2} = 0.5$
Now, $f(0.5) = -0.625$
Here $f(0.5) = -0.625$ and $f(0) = -1$ have same sign.
Therefore, we replace 0 by 0.5 in the current interval $(0, 1)$ to obtain new interval of approximation as given below

$$(0.5, 1)$$

Let $X_2 = \dfrac{0.5+1}{2} = 0.75$

Now, $f(0.75) = -0.015625$
Here $f(0.75) = -0.015625$ and $f(0.5) = -0.625$ have same sign.
Therefore, we replace 0.5 by 0.75 in the current interval $(0.5, 1)$ to obtain new interval of approximation as given below

$$(0.75, 1)$$

Let $X_3 = \dfrac{0.75+1}{2} = 0.875$

Now, $f(0.875) = 0.435546875$
Here $f(0.875) = 0.435546875$ and $f(1) = 1$ have same sign.

Therefore, we replace 1 by 0.875 in the current interval $(0.75, 1)$ to obtain new interval of approximation as given below

$$(0.75, 0.875)$$

Let $X_4 = \dfrac{0.75 + 0.875}{2} = 0.8125$

Now, $f(0.8125) = 0.196533203125$
Here $f(0.8125) = 0.196533203125$ and $f(0.875) = 0.435546875$ have same sign.
Therefore, we replace 0.875 by 0.8125 in the current interval $(0.75, 0.875)$ to obtain new interval of approximation as given below

$$(0.75, 0.8125)$$

Let $X_5 = \dfrac{0.75 + 0.8125}{2} = 0.78125$

Now, $f(0.78125) = 0.0871887207031$
Here $f(0.78125) = 0.0871887207031$ and $f(0.8125) = 0.196533203125$ have same sign.
Therefore, we replace 0.8125 by 0.78125 in the current interval $(0.75, 0.8125)$ to obtain new interval of approximation as given below

$$(0.75, 0.78125)$$

Let $X_6 = \dfrac{0.75 + 0.78125}{2} = 0.765625$

Now, $f(0.765625) = 0.0349769592285$
Here $f(0.765625) = 0.0349769592285$ and $f(0.78125) = 0.0871887207031$ have same sign.
Therefore, we replace 0.78125 by 0.765625 in the current interval $(0.75, 0.78125)$ to obtain new interval of approximation as given below

$$(0.75, 0.765625)$$

Let $X_7 = \dfrac{0.75 + 0.765625}{2} = 0.7578125$

Now, $f(0.7578125) = 0.00947618484497$
Here $f(0.7578125) = 0.00947618484497$ and $f(0.765625) = 0.0349769592285$ have same sign.
Therefore, we replace 0.765625 by 0.7578125 in the current interval $(0.75, 0.765625)$ to obtain new interval of approximation as given below

$$(0.75, 0.7578125)$$

Let $X_8 = \dfrac{0.75 + 0.7578125}{2} = 0.75390625$

Now, $f(0.75390625) = -0.0031241774559$
Here $f(0.75390625) = -0.0031241774559$ and $f(0.75) = -0.015625$ have same sign.
Therefore, we replace 0.75 by 0.75390625 in the current interval $(0.75, 0.7578125)$ to obtain new interval of approximation as given below

$$(0.75390625, 0.7578125)$$

Let $X_9 = \dfrac{0.75390625 + 0.7578125}{2} = 0.755859375$

Now, $f(0.755859375) = 0.0031635388732$
Here $f(0.755859375) = 0.0031635388732$ and $f(0.7578125) = 0.00947618484497$ have same sign.

Therefore, we replace 0.7578125 by 0.755859375 in the current interval $(0.75390625, 0.7578125)$ to obtain new interval of approximation as given below

$$(0.75390625, 0.755859375)$$

Let $X_{10} = \dfrac{0.75390625 + 0.755859375}{2} = 0.7548828125$

Now, $f(0.7548828125) = 1.65672972798e - 005$
Here $f(0.7548828125) = 1.65672972798e - 005$ and $f(0.755859375) = 0.0031635388732$ have same sign.
Therefore, we replace 0.755859375 by 0.7548828125 in the current interval $(0.75390625, 0.755859375)$ to obtain new interval of approximation as given below

$$(0.75390625, 0.7548828125)$$

Let $X_{11} = \dfrac{0.75390625 + 0.7548828125}{2} = 0.75439453125$

We find that 3 digits immediately after the decimal point in x_{10} and x_{11} are same.

Therefore 0.754 is an approximate real root of $x^3 + x^2 - 1 = 0$ correct upto 3 decimal places.

3. Using Bisection method find a real root of the equation $x^3 - x - 1 = 0$ correct upto three decimal palaces

Solution:
For $f(x) = x^3 - x - 1$, we have $f(1) = -1$ and $f(2) = 5$.
Therefore, a root of the equation $x^3 - x - 1 = 0$ lies in $(1, 2)$

Let $X_1 = \dfrac{1+2}{2} = 1.5$
Now, $f(1.5) = 0.875$
Here $f(1.5) = 0.875$ and $f(2) = 5$ have same sign.
Therefore, we replace 2 by 1.5 in the current interval $(1, 2)$ to obtain new interval of approximation as given below

$$(1, 1.5)$$

Let $X_2 = \dfrac{1 + 1.5}{2} = 1.25$

Now, $f(1.25) = -0.296875$
Here $f(1.25) = -0.296875$ and $f(1) = -1$ have same sign.
Therefore, we replace 1 by 1.25 in the current interval $(1, 1.5)$ to obtain new interval of approximation as given below

$$(1.25, 1.5)$$

Let $X_3 = \dfrac{1.25 + 1.5}{2} = 1.375$

Now, $f(1.375) = 0.224609375$
Here $f(1.375) = 0.224609375$ and $f(1.5) = 0.875$ have same sign.

Therefore, we replace 1.5 by 1.375 in the current interval $(1.25, 1.5)$ to obtain new interval of approximation as given below

$$(1.25, 1.375)$$

Let $X_4 = \dfrac{1.25 + 1.375}{2} = 1.3125$

Now, $f(1.3125) = -0.051513671875$
Here $f(1.3125) = -0.051513671875$ and $f(1.25) = -0.296875$ have same sign.
Therefore, we replace 1.25 by 1.3125 in the current interval $(1.25, 1.375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.375)$$

Let $X_5 = \dfrac{1.3125 + 1.375}{2} = 1.34375$

Now, $f(1.34375) = 0.0826110839844$
Here $f(1.34375) = 0.0826110839844$ and $f(1.375) = 0.224609375$ have same sign.
Therefore, we replace 1.375 by 1.34375 in the current interval $(1.3125, 1.375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.34375)$$

Let $X_6 = \dfrac{1.3125 + 1.34375}{2} = 1.328125$

Now, $f(1.328125) = 0.014575958252$
Here $f(1.328125) = 0.014575958252$ and $f(1.34375) = 0.0826110839844$ have same sign.
Therefore, we replace 1.34375 by 1.328125 in the current interval $(1.3125, 1.34375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.328125)$$

Let $X_7 = \dfrac{1.3125 + 1.328125}{2} = 1.3203125$

Now, $f(1.3203125) = -0.0187106132507$
Here $f(1.3203125) = -0.0187106132507$ and $f(1.3125) = -0.051513671875$ have same sign.
Therefore, we replace 1.3125 by 1.3203125 in the current interval $(1.3125, 1.328125)$ to obtain new interval of approximation as given below

$$(1.3203125, 1.328125)$$

Let $X_8 = \dfrac{1.3203125 + 1.328125}{2} = 1.32421875$

Now, $f(1.32421875) = -0.00212794542313$
Here $f(1.32421875) = -0.00212794542313$ and $f(1.3203125) = -0.0187106132507$ have same sign.
Therefore, we replace 1.3203125 by 1.32421875 in the current interval $(1.3203125, 1.328125)$ to obtain new interval of approximation as given below

$$(1.32421875, 1.328125)$$

Let $X_9 = \dfrac{1.32421875 + 1.328125}{2} = 1.326171875$

Now, $f(1.326171875) = 0.00620882958174$
Here $f(1.326171875) = 0.00620882958174$ and $f(1.328125) = 0.014575958252$ have same sign.

Therefore, we replace 1.328125 by 1.326171875 in the current interval $(1.32421875, 1.328125)$ to obtain new interval of approximation as given below

$$(1.32421875, 1.326171875)$$

Let $X_{10} = \dfrac{1.32421875 + 1.326171875}{2} = 1.3251953125$

Now, $f(1.3251953125) = 0.0020366506651$
Here $f(1.3251953125) = 0.0020366506651$ and $f(1.326171875) = 0.00620882958174$ have same sign.
Therefore, we replace 1.326171875 by 1.3251953125 in the current interval $(1.32421875, 1.326171875)$ to obtain new interval of approximation as given below

$$(1.32421875, 1.3251953125)$$

Let $X_{11} = \dfrac{1.32421875 + 1.3251953125}{2} = 1.32470703125$

Now, $f(1.32470703125) = -4.65948833153e - 005$
Here $f(1.32470703125) = -4.65948833153e - 005$ and $f(1.32421875) = -0.00212794542313$ have same sign.
Therefore, we replace 1.32421875 by 1.32470703125 in the current interval $(1.32421875, 1.3251953125)$ to obtain new interval of approximation as given below

$$(1.32470703125, 1.3251953125)$$

Let $X_{12} = \dfrac{1.32470703125 + 1.3251953125}{2} = 1.32495117188$

We find that 3 digits immediately after the decimal point in x_{11} and x_{12} are same.

Therefore 1.324 is an approximate real root of $x^3 - x - 1 = 0$ correct upto 3 decimal places.

4. **Using Bisection method find a real root of the equation $x^3 - 4x - 9 = 0$ correct upto three decimal palaces**

Solution:
For $f(x) = x^3 - 4x - 9$, we have $f(2) = -9$ and $f(3) = 6$.
Therefore, a root of the equation $x^3 - 4x - 9 = 0$ lies in $(2, 3)$

Let $X_1 = \dfrac{2+3}{2} = 2.5$
Now, $f(2.5) = -3.375$
Here $f(2.5) = -3.375$ and $f(2) = -9$ have same sign.
Therefore, we replace 2 by 2.5 in the current interval $(2, 3)$ to obtain new interval of approximation as given below

$$(2.5, 3)$$

Let $X_2 = \dfrac{2.5+3}{2} = 2.75$

Now, $f(2.75) = 0.796875$

Here $f(2.75) = 0.796875$ and $f(3) = 6$ have same sign.
Therefore, we replace 3 by 2.75 in the current interval $(2.5, 3)$ to obtain new interval of approximation as given below

$$(2.5, 2.75)$$

Let $X_3 = \dfrac{2.5 + 2.75}{2} = 2.625$

Now, $f(2.625) = -1.412109375$
Here $f(2.625) = -1.412109375$ and $f(2.5) = -3.375$ have same sign.
Therefore, we replace 2.5 by 2.625 in the current interval $(2.5, 2.75)$ to obtain new interval of approximation as given below

$$(2.625, 2.75)$$

Let $X_4 = \dfrac{2.625 + 2.75}{2} = 2.6875$

Now, $f(2.6875) = -0.339111328125$
Here $f(2.6875) = -0.339111328125$ and $f(2.625) = -1.412109375$ have same sign.
Therefore, we replace 2.625 by 2.6875 in the current interval $(2.625, 2.75)$ to obtain new interval of approximation as given below

$$(2.6875, 2.75)$$

Let $X_5 = \dfrac{2.6875 + 2.75}{2} = 2.71875$

Now, $f(2.71875) = 0.220916748047$
Here $f(2.71875) = 0.220916748047$ and $f(2.75) = 0.796875$ have same sign.
Therefore, we replace 2.75 by 2.71875 in the current interval $(2.6875, 2.75)$ to obtain new interval of approximation as given below

$$(2.6875, 2.71875)$$

Let $X_6 = \dfrac{2.6875 + 2.71875}{2} = 2.703125$

Now, $f(2.703125) = -0.0610771179199$
Here $f(2.703125) = -0.0610771179199$ and $f(2.6875) = -0.339111328125$ have same sign.
Therefore, we replace 2.6875 by 2.703125 in the current interval $(2.6875, 2.71875)$ to obtain new interval of approximation as given below

$$(2.703125, 2.71875)$$

Let $X_7 = \dfrac{2.703125 + 2.71875}{2} = 2.7109375$

Now, $f(2.7109375) = 0.0794234275818$
Here $f(2.7109375) = 0.0794234275818$ and $f(2.71875) = 0.220916748047$ have same sign.
Therefore, we replace 2.71875 by 2.7109375 in the current interval $(2.703125, 2.71875)$ to obtain new interval of approximation as given below

$$(2.703125, 2.7109375)$$

Let $X_8 = \dfrac{2.703125 + 2.7109375}{2} = 2.70703125$

Now, $f(2.70703125) = 0.00904923677444$
Here $f(2.70703125) = 0.00904923677444$ and $f(2.7109375) = 0.0794234275818$ have same sign.

Therefore, we replace 2.7109375 by 2.70703125 in the current interval $(2.703125, 2.7109375)$ to obtain new interval of approximation as given below

$$(2.703125, 2.70703125)$$

Let $X_9 = \dfrac{2.703125 + 2.70703125}{2} = 2.705078125$

Now, $f(2.705078125) = -0.0260448977351$
Here $f(2.705078125) = -0.0260448977351$ and $f(2.703125) = -0.0610771179199$ have same sign.
Therefore, we replace 2.703125 by 2.705078125 in the current interval $(2.703125, 2.70703125)$ to obtain new interval of approximation as given below

$$(2.705078125, 2.70703125)$$

Let $X_{10} = \dfrac{2.705078125 + 2.70703125}{2} = 2.7060546875$

Now, $f(2.7060546875) = -0.0085055725649$
Here $f(2.7060546875) = -0.0085055725649$ and $f(2.705078125) = -0.0260448977351$ have same sign.
Therefore, we replace 2.705078125 by 2.7060546875 in the current interval $(2.705078125, 2.70703125)$ to obtain new interval of approximation as given below

$$(2.7060546875, 2.70703125)$$

Let $X_{11} = \dfrac{2.7060546875 + 2.70703125}{2} = 2.70654296875$

We find that 3 digits immediately after the decimal point in x_{10} and x_{11} are same.

Therefore 2.706 is an approximate real root of $x^3 - 4x - 9 = 0$ correct upto 3 decimal places.

5. **Using Bisection method find a real root of the equation $x^3 - x - 4 = 0$ correct upto three decimal palaces**

Solution:
For $f(x) = x^3 - x - 4$, we have $f(1) = -4$ and $f(2) = 2$.
Therefore, a root of the equation $x^3 - x - 4 = 0$ lies in $(1, 2)$

Let $X_1 = \dfrac{1+2}{2} = 1.5$
Now, $f(1.5) = -2.125$
Here $f(1.5) = -2.125$ and $f(1) = -4$ have same sign.
Therefore, we replace 1 by 1.5 in the current interval $(1, 2)$ to obtain new interval of approximation as given below

$$(1.5, 2)$$

Let $X_2 = \dfrac{1.5 + 2}{2} = 1.75$

Now, $f(1.75) = -0.390625$

Here $f(1.75) = -0.390625$ and $f(1.5) = -2.125$ have same sign.
Therefore, we replace 1.5 by 1.75 in the current interval $(1.5, 2)$ to obtain new interval of approximation as given below

$$(1.75, 2)$$

Let $X_3 = \dfrac{1.75 + 2}{2} = 1.875$

Now, $f(1.875) = 0.716796875$
Here $f(1.875) = 0.716796875$ and $f(2) = 2$ have same sign.
Therefore, we replace 2 by 1.875 in the current interval $(1.75, 2)$ to obtain new interval of approximation as given below

$$(1.75, 1.875)$$

Let $X_4 = \dfrac{1.75 + 1.875}{2} = 1.8125$

Now, $f(1.8125) = 0.141845703125$
Here $f(1.8125) = 0.141845703125$ and $f(1.875) = 0.716796875$ have same sign.
Therefore, we replace 1.875 by 1.8125 in the current interval $(1.75, 1.875)$ to obtain new interval of approximation as given below

$$(1.75, 1.8125)$$

Let $X_5 = \dfrac{1.75 + 1.8125}{2} = 1.78125$

Now, $f(1.78125) = -0.129608154297$
Here $f(1.78125) = -0.129608154297$ and $f(1.75) = -0.390625$ have same sign.
Therefore, we replace 1.75 by 1.78125 in the current interval $(1.75, 1.8125)$ to obtain new interval of approximation as given below

$$(1.78125, 1.8125)$$

Let $X_6 = \dfrac{1.78125 + 1.8125}{2} = 1.796875$

Now, $f(1.796875) = 0.00480270385742$
Here $f(1.796875) = 0.00480270385742$ and $f(1.8125) = 0.141845703125$ have same sign.
Therefore, we replace 1.8125 by 1.796875 in the current interval $(1.78125, 1.8125)$ to obtain new interval of approximation as given below

$$(1.78125, 1.796875)$$

Let $X_7 = \dfrac{1.78125 + 1.796875}{2} = 1.7890625$

Now, $f(1.7890625) = -0.0627303123474$
Here $f(1.7890625) = -0.0627303123474$ and $f(1.78125) = -0.129608154297$ have same sign.
Therefore, we replace 1.78125 by 1.7890625 in the current interval $(1.78125, 1.796875)$ to obtain new interval of approximation as given below

$$(1.7890625, 1.796875)$$

Let $X_8 = \dfrac{1.7890625 + 1.796875}{2} = 1.79296875$

Now, $f(1.79296875) = -0.0290458798409$
Here $f(1.79296875) = -0.0290458798409$ and $f(1.7890625) = -0.0627303123474$ have same

sign.
Therefore, we replace 1.7890625 by 1.79296875 in the current interval $(1.7890625, 1.796875)$ to obtain new interval of approximation as given below

$$(1.79296875, 1.796875)$$

Let $X_9 = \dfrac{1.79296875 + 1.796875}{2} = 1.794921875$

Now, $f(1.794921875) = -0.0121421292424$
Here $f(1.794921875) = -0.0121421292424$ and $f(1.79296875) = -0.0290458798409$ have same sign.
Therefore, we replace 1.79296875 by 1.794921875 in the current interval $(1.79296875, 1.796875)$ to obtain new interval of approximation as given below

$$(1.794921875, 1.796875)$$

Let $X_{10} = \dfrac{1.794921875 + 1.796875}{2} = 1.7958984375$

Now, $f(1.7958984375) = -0.00367485079914$
Here $f(1.7958984375) = -0.00367485079914$ and $f(1.794921875) = -0.0121421292424$ have same sign.
Therefore, we replace 1.794921875 by 1.7958984375 in the current interval $(1.794921875, 1.796875)$ to obtain new interval of approximation as given below

$$(1.7958984375, 1.796875)$$

Let $X_{11} = \dfrac{1.7958984375 + 1.796875}{2} = 1.79638671875$

Now, $f(1.79638671875) = 0.000562641653232$
Here $f(1.79638671875) = 0.000562641653232$ and $f(1.796875) = 0.00480270385742$ have same sign.
Therefore, we replace 1.796875 by 1.79638671875 in the current interval $(1.7958984375, 1.796875)$ to obtain new interval of approximation as given below

$$(1.7958984375, 1.79638671875)$$

Let $X_{12} = \dfrac{1.7958984375 + 1.79638671875}{2} = 1.79614257813$

We find that 3 digits immediately after the decimal point in x_{11} and x_{12} are same.

Therefore 1.796 is an approximate real root of $x^3 - x - 4 = 0$ correct upto 3 decimal places.

6. Using Bisection method find a real root of the equation $x^3 - 10x + 3 = 0$ correct upto four decimal palaces

Solution:
For $f(x) = x^3 - 10x + 3$, we have $f(0) = 3$ and $f(1) = -6$.
Therefore, a root of the equation $x^3 - 10x + 3 = 0$ lies in $(0, 1)$

Let $X_1 = \dfrac{0+1}{2} = 0.5$
Now, $f(0.5) = -1.875$
Here $f(0.5) = -1.875$ and $f(1) = -6$ have same sign.
Therefore, we replace 1 by 0.5 in the current interval $(0, 1)$ to obtain new interval of approximation as given below

$$(0, 0.5)$$

Let $X_2 = \dfrac{0+0.5}{2} = 0.25$

Now, $f(0.25) = 0.515625$
Here $f(0.25) = 0.515625$ and $f(0) = 3$ have same sign.
Therefore, we replace 0 by 0.25 in the current interval $(0, 0.5)$ to obtain new interval of approximation as given below

$$(0.25, 0.5)$$

Let $X_3 = \dfrac{0.25+0.5}{2} = 0.375$

Now, $f(0.375) = -0.697265625$
Here $f(0.375) = -0.697265625$ and $f(0.5) = -1.875$ have same sign.
Therefore, we replace 0.5 by 0.375 in the current interval $(0.25, 0.5)$ to obtain new interval of approximation as given below

$$(0.25, 0.375)$$

Let $X_4 = \dfrac{0.25+0.375}{2} = 0.3125$

Now, $f(0.3125) = -0.094482421875$
Here $f(0.3125) = -0.094482421875$ and $f(0.375) = -0.697265625$ have same sign.
Therefore, we replace 0.375 by 0.3125 in the current interval $(0.25, 0.375)$ to obtain new interval of approximation as given below

$$(0.25, 0.3125)$$

Let $X_5 = \dfrac{0.25+0.3125}{2} = 0.28125$

Now, $f(0.28125) = 0.209747314453$
Here $f(0.28125) = 0.209747314453$ and $f(0.25) = 0.515625$ have same sign.
Therefore, we replace 0.25 by 0.28125 in the current interval $(0.25, 0.3125)$ to obtain new interval of approximation as given below

$$(0.28125, 0.3125)$$

Let $X_6 = \dfrac{0.28125+0.3125}{2} = 0.296875$

Now, $f(0.296875) = 0.0574150085449$
Here $f(0.296875) = 0.0574150085449$ and $f(0.28125) = 0.209747314453$ have same sign.
Therefore, we replace 0.28125 by 0.296875 in the current interval $(0.28125, 0.3125)$ to obtain new interval of approximation as given below

$$(0.296875, 0.3125)$$

Let $X_7 = \dfrac{0.296875 + 0.3125}{2} = 0.3046875$

Now, $f(0.3046875) = -0.0185894966125$
Here $f(0.3046875) = -0.0185894966125$ and $f(0.3125) = -0.094482421875$ have same sign.
Therefore, we replace 0.3125 by 0.3046875 in the current interval $(0.296875, 0.3125)$ to obtain new interval of approximation as given below

$$(0.296875, 0.3046875)$$

Let $X_8 = \dfrac{0.296875 + 0.3046875}{2} = 0.30078125$

Now, $f(0.30078125) = 0.0193989872932$
Here $f(0.30078125) = 0.0193989872932$ and $f(0.296875) = 0.0574150085449$ have same sign.
Therefore, we replace 0.296875 by 0.30078125 in the current interval $(0.296875, 0.3046875)$ to obtain new interval of approximation as given below

$$(0.30078125, 0.3046875)$$

Let $X_9 = \dfrac{0.30078125 + 0.3046875}{2} = 0.302734375$

Now, $f(0.302734375) = 0.00040128082037$
Here $f(0.302734375) = 0.00040128082037$ and $f(0.30078125) = 0.0193989872932$ have same sign.
Therefore, we replace 0.30078125 by 0.302734375 in the current interval $(0.30078125, 0.3046875)$ to obtain new interval of approximation as given below

$$(0.302734375, 0.3046875)$$

Let $X_{10} = \dfrac{0.302734375 + 0.3046875}{2} = 0.3037109375$

Now, $f(0.3037109375) = -0.00909497682005$
Here $f(0.3037109375) = -0.00909497682005$ and $f(0.3046875) = -0.0185894966125$ have same sign.
Therefore, we replace 0.3046875 by 0.3037109375 in the current interval $(0.302734375, 0.3046875)$ to obtain new interval of approximation as given below

$$(0.302734375, 0.3037109375)$$

Let $X_{11} = \dfrac{0.302734375 + 0.3037109375}{2} = 0.30322265625$

Now, $f(0.30322265625) = -0.00434706488159$
Here $f(0.30322265625) = -0.00434706488159$ and $f(0.3037109375) = -0.00909497682005$ have same sign.
Therefore, we replace 0.3037109375 by 0.30322265625 in the current interval $(0.302734375, 0.3037109375)$ to obtain new interval of approximation as given below

$$(0.302734375, 0.30322265625)$$

Let $X_{12} = \dfrac{0.302734375 + 0.30322265625}{2} = 0.302978515625$

Now, $f(0.302978515625) = -0.00197294620739$
Here $f(0.302978515625) = -0.00197294620739$ and $f(0.30322265625) = -0.00434706488159$ have same sign.
Therefore, we replace 0.30322265625 by 0.302978515625 in the current interval $(0.302734375, 0.30322265625)$

to obtain new interval of approximation as given below

$$(0.302734375, 0.302978515625)$$

Let $X_{13} = \dfrac{0.302734375 + 0.302978515625}{2} = 0.302856445313$

Now, $f(0.302856445313) = -0.000785846232247$
Here $f(0.302856445313) = -0.000785846232247$ and $f(0.302978515625) = -0.00197294620739$ have same sign.
Therefore, we replace 0.302978515625 by 0.302856445313 in the current interval $(0.302734375, 0.302978515625)$ to obtain new interval of approximation as given below

$$(0.302734375, 0.302856445313)$$

Let $X_{14} = \dfrac{0.302734375 + 0.302856445313}{2} = 0.302795410156$

Now, $f(0.302795410156) = -0.000192286089941$
Here $f(0.302795410156) = -0.000192286089941$ and $f(0.302856445313) = -0.000785846232247$ have same sign.
Therefore, we replace 0.302856445313 by 0.302795410156 in the current interval $(0.302734375, 0.302856445313)$ to obtain new interval of approximation as given below

$$(0.302734375, 0.302795410156)$$

Let $X_{15} = \dfrac{0.302734375 + 0.302795410156}{2} = 0.302764892578$

We find that 4 digits immediately after the decimal point in x_{14} and x_{15} are same.

Therefore 0.3027 is an approximate real root of $x^3 - 10x + 3 = 0$ correct upto 4 decimal places.

7. Using Bisection method find a real root of the equation $5e^{-x} - x = 0$ correct upto two decimal palaces

Solution:
For $f(x) = 5e^{-x} - x$, we have $f(1) = 0.008393972$ and $f(2) = -1.323324$.
Therefore, a root of the equation $5e^{-x} - x = 0$ lies in $(1, 2)$

Let $X_1 = \dfrac{1+2}{2} = 1.5$
Now, $f(1.5) = -0.384349199258$
Here $f(1.5) = -0.384349199258$ and $f(2) = -1.32332358382$ have same sign.
Therefore, we replace 2 by 1.5 in the current interval $(1, 2)$ to obtain new interval of approximation as given below

$$(1, 1.5)$$

Let $X_2 = \dfrac{1+1.5}{2} = 1.25$

Now, $f(1.25) = 0.182523984301$
Here $f(1.25) = 0.182523984301$ and $f(1) = 0.839397205857$ have same sign.

Therefore, we replace 1 by 1.25 in the current interval $(1, 1.5)$ to obtain new interval of approximation as given below

$$(1.25, 1.5)$$

Let $X_3 = \dfrac{1.25 + 1.5}{2} = 1.375$

Now, $f(1.375) = -0.110802020976$
Here $f(1.375) = -0.110802020976$ and $f(1.5) = -0.384349199258$ have same sign.
Therefore, we replace 1.5 by 1.375 in the current interval $(1.25, 1.5)$ to obtain new interval of approximation as given below

$$(1.25, 1.375)$$

Let $X_4 = \dfrac{1.25 + 1.375}{2} = 1.3125$

Now, $f(1.3125) = 0.0332317436459$
Here $f(1.3125) = 0.0332317436459$ and $f(1.25) = 0.182523984301$ have same sign.
Therefore, we replace 1.25 by 1.3125 in the current interval $(1.25, 1.375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.375)$$

Let $X_5 = \dfrac{1.3125 + 1.375}{2} = 1.34375$

Now, $f(1.34375) = -0.0394220693686$
Here $f(1.34375) = -0.0394220693686$ and $f(1.375) = -0.110802020976$ have same sign.
Therefore, we replace 1.375 by 1.34375 in the current interval $(1.3125, 1.375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.34375)$$

Let $X_6 = \dfrac{1.3125 + 1.34375}{2} = 1.328125$

Now, $f(1.328125) = -0.00325689321552$
Here $f(1.328125) = -0.00325689321552$ and $f(1.34375) = -0.0394220693686$ have same sign.
Therefore, we replace 1.34375 by 1.328125 in the current interval $(1.3125, 1.34375)$ to obtain new interval of approximation as given below

$$(1.3125, 1.328125)$$

Let $X_7 = \dfrac{1.3125 + 1.328125}{2} = 1.3203125$

We find that 2 digits immediately after the decimal point in x_6 and x_7 are same.

Therefore 1.32 is an approximate real root of $5e^{-x} - x = 0$ correct upto 2 decimal places.

8. **Using Bisection method find a real root of the equation $2x\log_{10}(x+5) - 6 = 0$ correct upto three decimal palaces**

Solution:
For $f(x) = 2x\log_{10}(x+5) - 6$, we have $f(3) = -0.005814601$ and $f(4) = 1.633940$.
Therefore, a root of the equation $2x\log_{10}(x+5) - 6 = 0$ lies in $(3, 4)$

Let $X_1 = \dfrac{3+4}{2} = 3.5$
Now, $f(3.5) = 0.50593248$
Here $f(3.5) = 0.50593248$ and $f(4) = 1.63394007551$ have same sign.
Therefore, we replace 4 by 3.5 in the current interval $(3, 4)$ to obtain new interval of approximation as given below

$$(3, 3.5)$$

Let $X_2 = \dfrac{3+3.5}{2} = 3.25$

Now, $f(3.25) = -0.0430493344255$
Here $f(3.25) = -0.0430493344255$ and $f(3) = -0.581460078048$ have same sign.
Therefore, we replace 3 by 3.25 in the current interval $(3, 3.5)$ to obtain new interval of approximation as given below

$$(3.25, 3.5)$$

Let $X_3 = \dfrac{3.25+3.5}{2} = 3.375$

Now, $f(3.375) = 0.230147506035$
Here $f(3.375) = 0.230147506035$ and $f(3.5) = 0.50593248$ have same sign.
Therefore, we replace 3.5 by 3.375 in the current interval $(3.25, 3.5)$ to obtain new interval of approximation as given below

$$(3.25, 3.375)$$

Let $X_4 = \dfrac{3.25+3.375}{2} = 3.3125$

Now, $f(3.3125) = 0.0932222363114$
Here $f(3.3125) = 0.0932222363114$ and $f(3.375) = 0.230147506035$ have same sign.
Therefore, we replace 3.375 by 3.3125 in the current interval $(3.25, 3.375)$ to obtain new interval of approximation as given below

$$(3.25, 3.3125)$$

Let $X_5 = \dfrac{3.25+3.3125}{2} = 3.28125$

Now, $f(3.28125) = 0.0250043149859$
Here $f(3.28125) = 0.0250043149859$ and $f(3.3125) = 0.0932222363114$ have same sign.
Therefore, we replace 3.3125 by 3.28125 in the current interval $(3.25, 3.3125)$ to obtain new interval of approximation as given below

$$(3.25, 3.28125)$$

Let $X_6 = \dfrac{3.25+3.28125}{2} = 3.265625$

Now, $f(3.265625) = -0.00904309710246$
Here $f(3.265625) = -0.00904309710246$ and $f(3.25) = -0.0430493344255$ have same sign.
Therefore, we replace 3.25 by 3.265625 in the current interval $(3.25, 3.28125)$ to obtain new

interval of approximation as given below

$$(3.265625, 3.28125)$$

$$\text{Let } X_7 = \frac{3.265625 + 3.28125}{2} = 3.2734375$$

Now, $f(3.2734375) = 0.0079754687903$
Here $f(3.2734375) = 0.0079754687903$ and $f(3.28125) = 0.0250043149859$ have same sign.
Therefore, we replace 3.28125 by 3.2734375 in the current interval $(3.265625, 3.28125)$ to obtain new interval of approximation as given below

$$(3.265625, 3.2734375)$$

$$\text{Let } X_8 = \frac{3.265625 + 3.2734375}{2} = 3.26953125$$

Now, $f(3.26953125) = -0.000535100029455$
Here $f(3.26953125) = -0.000535100029455$ and $f(3.265625) = -0.00904309710246$ have same sign.
Therefore, we replace 3.265625 by 3.26953125 in the current interval $(3.265625, 3.2734375)$ to obtain new interval of approximation as given below

$$(3.26953125, 3.2734375)$$

$$\text{Let } X_9 = \frac{3.26953125 + 3.2734375}{2} = 3.271484375$$

Now, $f(3.271484375) = 0.0037198630166$
Here $f(3.271484375) = 0.0037198630166$ and $f(3.2734375) = 0.0079754687903$ have same sign.
Therefore, we replace 3.2734375 by 3.271484375 in the current interval $(3.26953125, 3.2734375)$ to obtain new interval of approximation as given below

$$(3.26953125, 3.271484375)$$

$$\text{Let } X_{10} = \frac{3.26953125 + 3.271484375}{2} = 3.2705078125$$

Now, $f(3.2705078125) = 0.00159230113955$
Here $f(3.2705078125) = 0.00159230113955$ and $f(3.271484375) = 0.0037198630166$ have same sign.
Therefore, we replace 3.271484375 by 3.2705078125 in the current interval $(3.26953125, 3.271484375)$ to obtain new interval of approximation as given below

$$(3.26953125, 3.2705078125)$$

$$\text{Let } X_{11} = \frac{3.26953125 + 3.2705078125}{2} = 3.27001953125$$

We find that 3 digits immediately after the decimal point in x_{10} and x_{11} are same.

Therefore 3.270 is an approximate real root of $2x\log_{10}(x+5) - 6 = 0$ correct upto 3 decimal places.

9. Discuss the method of False Position for approximation of a root of an equation.

Answer:
Suppose, for the equation $f(x) = 0$ the function $f(x)$ is continuous on some interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs.
Therefore there is some real root, say ξ, of the equation in $[a, b]$.

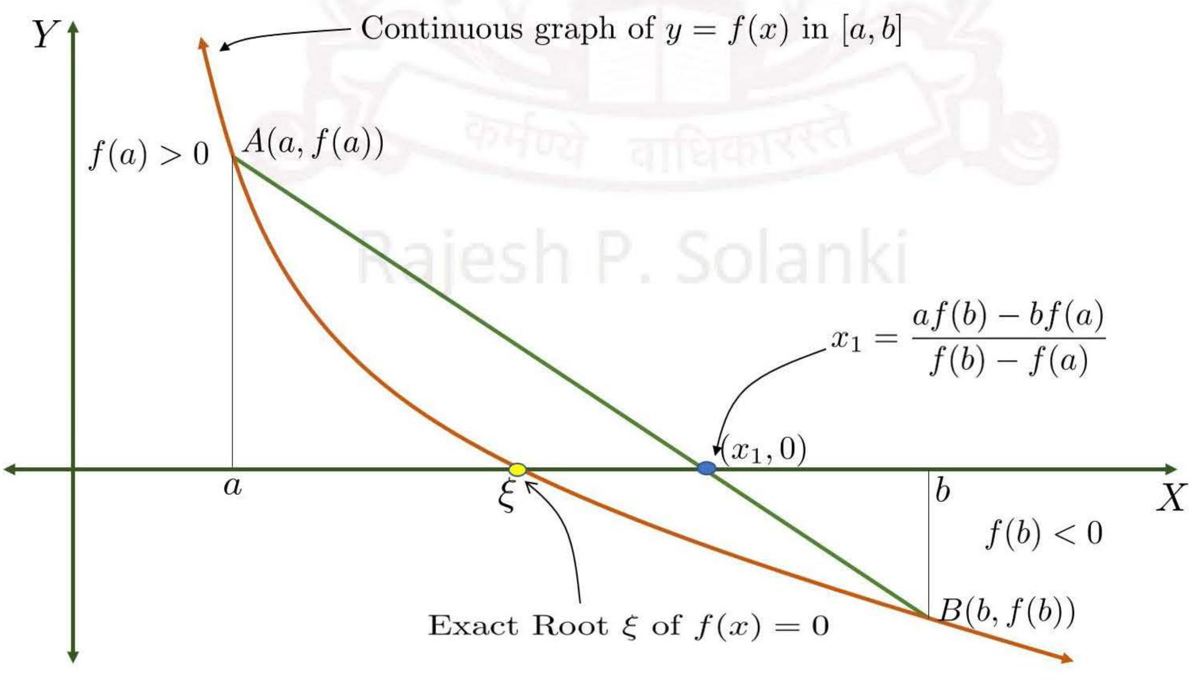
Recall that the equation of a line or line segment joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

So, the equation of the chord $\overline{AB}$ of the curve $y = f(x)$ joining the end points $A(a, f(a))$ and $B(b, f(b))$ is given by

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

Suppose, the point of intersection of the chord $\overline{AB}$ and the X-axis is $(x_1, 0)$. As the point of



Approximation of root of an equation $f(x) = 0$ using False Position Method intersection satisfies the equation we get,

$$\frac{0-f(a)}{x_1-a}=\frac{f(b)-f(a)}{b-a}$$
$$\therefore -\frac{x_1-a}{f(a)}=\frac{b-a}{f(b)-f(a)}$$
$$\therefore x_1-a=-f(a)\frac{b-a}{f(b)-f(a)}$$
$$\therefore x_1=a-f(a)\frac{b-a}{f(b)-f(a)}$$
$$\therefore x_1=\frac{af(b)-af(a)-bf(a)+af(a)}{f(b)-f(a)}$$
$$\therefore x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

Therefore, the x-coordinate of the point of intersection is given by

$$x_1=\frac{af(b)-bf(a)}{f(b)-f(a)}$$

which lies in between a and b and can be treated as an approximation of actual real root.

Now, if $f(x_1)$ and $f(a)$ (or $f(b)$) have the same sign then $f(x_1)$ and $f(b)$ (or $f(a)$) are of opposite signs. Therefore the actual root ξ must be in $[x_1, b]$ (or $[a, x_1]$).

Once more we obtain next approximation x_2 as the x-coordinate of the point where the chord joining the end-points of the curve on $[x_1, b]$ (or $[a, x_1]$) intersects the X-axis.

Continuing similarly we get a sequance of approximations $x_1, x_2, x_3, \ldots$ shch that the approximations get closer and closed to ξ (because successively subinterval lengths are becoming smaller and smaller.)

If we want to approximate ξ correct upto m decimal places then we shall stop at a stage where for some positive integer n we find that the first m digits immediately after the decimal points in x_n adn x_{n+1} are identical.

In that case we can say that x_n is correct upto m decimal places.

10. **Find a real root of $x^3+x^2+2x-1=0$ by method of False Position correct upto three decimal places**

Solution:
For $f(x)=x^3+x^2+2x-1$, we have $f(0)=-1$ and $f(1)=3$.
Therefore, a root of the equation $x^3+x^2+2x-1=0$ lies in $(0,1)$

Now, when an interval $[a,b]$ contains a root of the equatin we use **False Position** method formula $x_i=\dfrac{af(b)-bf(a)}{f(b)-f(a)}$ to obtain successive approximations as follows.

$$x_1 = \frac{(0)f(1) - (1)f(0)}{f(1) - f(0)}$$

$$= \frac{(0)(3) - (1)(-1)}{(3) - (-1)}$$

$$= 0.25$$

Now, $f(0.25) = -0.421875$
Here $f(0.25) = -0.421875$ and $f(0) = -1$ have same sign.
Therefore, we replace 0 by 0.25 in the current interval $(0, 1)$ to obtain new interval of approximation as given below

$$(0.25, 1)$$

$$x_2 = \frac{(0.25)f(1) - (1)f(0.25)}{f(1) - f(0.25)}$$

$$= \frac{(0.25)(3) - (1)(-0.421875)}{(3) - (-0.421875)}$$

$$= 0.342465753425$$

Now, $f(0.342465753425) = -0.157620361064$
Here $f(0.342465753425) = -0.157620361064$ and $f(0.25) = -0.421875$ have same sign.
Therefore, we replace 0.25 by 0.342465753425 in the current interval $(0.25, 1)$ to obtain new interval of approximation as given below

$$(0.342465753425, 1)$$

$$x_3 = \frac{(0.342465753425)f(1) - (1)f(0.342465753425)}{f(1) - f(0.342465753425)}$$

$$= \frac{(0.342465753425)(3) - (1)(-0.157620361064)}{(3) - (-0.157620361064)}$$

$$= 0.375288187253$$

Now, $f(0.375288187253) = -0.0557263545466$
Here $f(0.375288187253) = -0.0557263545466$ and $f(0.342465753425) = -0.157620361064$ have same sign.
Therefore, we replace 0.342465753425 by 0.375288187253 in the current interval $(0.342465753425, 1)$ to obtain new interval of approximation as given below

$$(0.375288187253, 1)$$

$$x_4 = \frac{(0.375288187253)f(1) - (1)f(0.375288187253)}{f(1) - f(0.375288187253)}$$

$$= \frac{(0.375288187253)(3) - (1)(-0.0557263545466)}{(3) - (-0.0557263545466)}$$

$$= 0.386680867071$$

Now, $f(0.386680867071) = -0.0192988403489$
Here $f(0.386680867071) = -0.0192988403489$ and $f(0.375288187253) = -0.0557263545466$ have same sign.
Therefore, we replace 0.375288187253 by 0.386680867071 in the current interval $(0.375288187253, 1)$ to obtain new interval of approximation as given below

$$(0.386680867071, 1)$$

$$x_5 = \frac{(0.386680867071)f(1) - (1)f(0.386680867071)}{f(1) - f(0.386680867071)}$$

$$= \frac{(0.386680867071)(3) - (1)(-0.0192988403489)}{(3) - (-0.0192988403489)}$$

$$= 0.390601097778$$

Now, $f(0.390601097778) = -0.00663488298198$
Here $f(0.390601097778) = -0.00663488298198$ and $f(0.386680867071) = -0.0192988403489$ have same sign.
Therefore, we replace 0.386680867071 by 0.390601097778 in the current interval $(0.386680867071, 1)$ to obtain new interval of approximation as given below

$$(0.390601097778, 1)$$

$$x_6 = \frac{(0.390601097778)f(1) - (1)f(0.390601097778)}{f(1) - f(0.390601097778)}$$

$$= \frac{(0.390601097778)(3) - (1)(-0.00663488298198)}{(3) - (-0.00663488298198)}$$

$$= 0.391945887074$$

Now, $f(0.391945887074) = -0.00227530164187$
Here $f(0.391945887074) = -0.00227530164187$ and $f(0.390601097778) = -0.00663488298198$ have same sign.
Therefore, we replace 0.390601097778 by 0.391945887074 in the current interval $(0.390601097778, 1)$ to obtain new interval of approximation as given below

$$(0.391945887074, 1)$$

$$x_7 = \frac{(0.391945887074)f(1) - (1)f(0.391945887074)}{f(1) - f(0.391945887074)}$$

$$= \frac{(0.391945887074)(3) - (1)(-0.00227530164187)}{(3) - (-0.00227530164187)}$$

$$= 0.392406706413$$

Now, $f(0.392406706413) = -0.000779592943718$
Here $f(0.392406706413) = -0.000779592943718$ and $f(0.391945887074) = -0.00227530164187$ have same sign.
Therefore, we replace 0.391945887074 by 0.392406706413 in the current interval $(0.391945887074, 1)$ to obtain new interval of approximation as given below

$$(0.392406706413, 1)$$

$$x_8 = \frac{(0.392406706413)f(1) - (1)f(0.392406706413)}{f(1) - f(0.392406706413)}$$

$$= \frac{(0.392406706413)(3) - (1)(-0.000779592943718)}{(3) - (-0.000779592943718)}$$

$$= 0.392564557208$$

We find that 3 digits immidiately after the decimal point in x_7 and x_8 are same.

Therefore 0.392 is an approximate real root of $x^3 + x^2 + 2x - 1 = 0$ correct upto 3 decimal places.

11. Find a real root of $x^3 - x - 4 = 0$ by method of False Position correct upto three decimal places

Solution:
For $f(x) = x^3 - x - 4$, we have $f(1) = -4$ and $f(2) = 2$.
Therefore, a root of the equation $x^3 - x - 4 = 0$ lies in $(1, 2)$

Now, when an interval $[a, b]$ contains a root of the equatin we use **False Position** method formula $x_i = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ to obtain successive approximations as follows.

$$x_1 = \frac{(1)f(2) - (2)f(1)}{f(2) - f(1)}$$

$$= \frac{(1)(2) - (2)(-4)}{(2) - (-4)}$$

$$= 1.66666666667$$

Now, $f(1.66666666667) = -1.03703703704$
Here $f(1.66666666667) = -1.03703703704$ and $f(1) = -4$ have same sign.
Therefore, we replace 1 by 1.66666666667 in the current interval $(1, 2)$ to obtain new interval of approximation as given below

$$(1.66666666667, 2)$$

$$x_2 = \frac{(1.66666666667)f(2) - (2)f(1.66666666667)}{f(2) - f(1.66666666667)}$$

$$= \frac{(1.66666666667)(2) - (2)(-1.03703703704)}{(2) - (-1.03703703704)}$$

$$= 1.78048780488$$

Now, $f(1.78048780488) = -0.136097851163$
Here $f(1.78048780488) = -0.136097851163$ and $f(1.66666666667) = -1.03703703704$ have same sign.
Therefore, we replace 1.66666666667 by 1.78048780488 in the current interval $(1.66666666667, 2)$ to obtain new interval of approximation as given below

$$(1.78048780488, 2)$$

$$x_3 = \frac{(1.78048780488)f(2) - (2)f(1.78048780488)}{f(2) - f(1.78048780488)}$$

$$= \frac{(1.78048780488)(2) - (2)(-0.136097851163)}{(2) - (-0.136097851163)}$$

$$= 1.79447365204$$

Now, $f(1.79447365204) = -0.0160250042084$
Here $f(1.79447365204) = -0.0160250042084$ and $f(1.78048780488) = -0.136097851163$ have same sign.
Therefore, we replace 1.78048780488 by 1.79447365204 in the current interval $(1.78048780488, 2)$ to obtain new interval of approximation as given below

$$(1.79447365204, 2)$$

$$x_4 = \frac{(1.79447365204)f(2) - (2)f(1.79447365204)}{f(2) - f(1.79447365204)}$$

$$= \frac{(1.79447365204)(2) - (2)(-0.0160250042084)}{(2) - (-0.0160250042084)}$$

$$= 1.79610734238$$

Now, $f(1.79610734238) = -0.00186220836725$
Here $f(1.79610734238) = -0.00186220836725$ and $f(1.79447365204) = -0.0160250042084$ have same sign.
Therefore, we replace 1.79447365204 by 1.79610734238 in the current interval $(1.79447365204, 2)$ to obtain new interval of approximation as given below

$$(1.79610734238, 2)$$

$$x_5 = \frac{(1.79610734238)f(2) - (2)f(1.79610734238)}{f(2) - f(1.79610734238)}$$

$$= \frac{(1.79610734238)(2) - (2)(-0.00186220836725)}{(2) - (-0.00186220836725)}$$

$$= 1.79629701109$$

We find that 3 digits immidiately after the decimal point in x_4 and x_5 are same.

Therefore 1.796 is an approximate real root of $x^3 - x - 4 = 0$ correct upto 3 decimal places.

12. Find a real root of $x^3 + x^2 - 10 = 0$ by method of False Position correct upto three decimal places

Solution:
For $f(x) = x^3 + x^2 - 10$, we have $f(1) = -8$ and $f(2) = 2$.
Therefore, a root of the equation $x^3 + x^2 - 10 = 0$ lies in $(1, 2)$

Now, when an interval $[a, b]$ contains a root of the equatin we use **False Position** method formula $x_i = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ to obtain successive approximations as follows.

$$x_1 = \frac{(1)f(2) - (2)f(1)}{f(2) - f(1)}$$

$$= \frac{(1)(2) - (2)(-8)}{(2) - (-8)}$$

$$= 1.8$$

Now, $f(1.8) = -0.928$
Here $f(1.8) = -0.928$ and $f(1) = -8$ have same sign.
Therefore, we replace 1 by 1.8 in the current interval $(1, 2)$ to obtain new interval of approximation as given below

$$(1.8, 2)$$

$$x_2 = \frac{(1.8)f(2) - (2)f(1.8)}{f(2) - f(1.8)}$$

$$= \frac{(1.8)(2) - (2)(-0.928)}{(2) - (-0.928)}$$

$$= 1.86338797814$$

Now, $f(1.86338797814) = -0.057702007037$
Here $f(1.86338797814) = -0.057702007037$ and $f(1.8) = -0.928$ have same sign.
Therefore, we replace 1.8 by 1.86338797814 in the current interval $(1.8, 2)$ to obtain new interval of approximation as given below

$$(1.86338797814, 2)$$

$$x_3 = \frac{(1.86338797814)f(2) - (2)f(1.86338797814)}{f(2) - f(1.86338797814)}$$

$$= \frac{(1.86338797814)(2) - (2)(-0.057702007037)}{(2) - (-0.057702007037)}$$

$$= 1.86721884764$$

Now, $f(1.86721884764) = -0.00342363937242$
Here $f(1.86721884764) = -0.00342363937242$ and $f(1.86338797814) = -0.057702007037$ have same sign.
Therefore, we replace 1.86338797814 by 1.86721884764 in the current interval $(1.86338797814, 2)$ to obtain new interval of approximation as given below

$$(1.86721884764, 2)$$

$$x_4 = \frac{(1.86721884764)f(2) - (2)f(1.86721884764)}{f(2) - f(1.86721884764)}$$

$$= \frac{(1.86721884764)(2) - (2)(-0.00342363937242)}{(2) - (-0.00342363937242)}$$

$$= 1.8674457566$$

We find that 3 digits immidiately after the decimal point in x_3 and x_4 are same.

Therefore 1.867 is an approximate real root of $x^3 + x^2 - 10 = 0$ correct upto 3 decimal places.

13. Find a real root of $x^3 - 2x - 5 = 0$ by method of False Position correct upto three decimal places

Solution:
For $f(x) = x^3 - 2x - 5$, we have $f(2) = -1$ and $f(3) = 16$.
Therefore, a root of the equation $x^3 - 2x - 5 = 0$ lies in $(2, 3)$

Now, when an interval $[a, b]$ contains a root of the equatin we use **False Position** method formula $x_i = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ to obtain successive approximations as follows.

$$x_1 = \frac{(2)f(3) - (3)f(2)}{f(3) - f(2)}$$

$$= \frac{(2)(16) - (3)(-1)}{(16) - (-1)}$$

$$= 2.05882352941$$

Now, $f(2.05882352941) = -0.390799918583$
Here $f(2.05882352941) = -0.390799918583$ and $f(2) = -1$ have same sign.
Therefore, we replace 2 by 2.05882352941 in the current interval $(2, 3)$ to obtain new interval of approximation as given below

$$(2.05882352941, 3)$$

$$x_2 = \frac{(2.05882352941)f(3) - (3)f(2.05882352941)}{f(3) - f(2.05882352941)}$$

$$= \frac{(2.05882352941)(16) - (3)(-0.390799918583)}{(16) - (-0.390799918583)}$$

$$= 2.08126365985$$

Now, $f(2.08126365985) = -0.147204059554$
Here $f(2.08126365985) = -0.147204059554$ and $f(2.05882352941) = -0.390799918583$ have same sign.
Therefore, we replace 2.05882352941 by 2.08126365985 in the current interval $(2.05882352941, 3)$ to obtain new interval of approximation as given below

$$(2.08126365985, 3)$$

$$x_3 = \frac{(2.08126365985)f(3) - (3)f(2.08126365985)}{f(3) - f(2.08126365985)}$$

$$= \frac{(2.08126365985)(16) - (3)(-0.147204059554)}{(16) - (-0.147204059554)}$$

$$= 2.08963921009$$

Now, $f(2.08963921009) = -0.0546765032733$
Here $f(2.08963921009) = -0.0546765032733$ and $f(2.08126365985) = -0.147204059554$ have same sign.
Therefore, we replace 2.08126365985 by 2.08963921009 in the current interval (2.08126365985, 3) to obtain new interval of approximation as given below

$$(2.08963921009, 3)$$

$$x_4 = \frac{(2.08963921009)f(3) - (3)f(2.08963921009)}{f(3) - f(2.08963921009)}$$

$$= \frac{(2.08963921009)(16) - (3)(-0.0546765032733)}{(16) - (-0.0546765032733)}$$

$$= 2.09273957432$$

Now, $f(2.09273957432) = -0.0202028663125$
Here $f(2.09273957432) = -0.0202028663125$ and $f(2.08963921009) = -0.0546765032733$ have same sign.
Therefore, we replace 2.08963921009 by 2.09273957432 in the current interval (2.08963921009, 3) to obtain new interval of approximation as given below

$$(2.09273957432, 3)$$

$$x_5 = \frac{(2.09273957432)f(3) - (3)f(2.09273957432)}{f(3) - f(2.09273957432)}$$

$$= \frac{(2.09273957432)(16) - (3)(-0.0202028663125)}{(16) - (-0.0202028663125)}$$

$$= 2.09388370846$$

Now, $f(2.09388370846) = -0.00745050593819$
Here $f(2.09388370846) = -0.00745050593819$ and $f(2.09273957432) = -0.0202028663125$ have same sign.
Therefore, we replace 2.09273957432 by 2.09388370846 in the current interval (2.09273957432, 3) to obtain new interval of approximation as given below

$$(2.09388370846, 3)$$

$$x_6 = \frac{(2.09388370846)f(3) - (3)f(2.09388370846)}{f(3) - f(2.09388370846)}$$

$$= \frac{(2.09388370846)(16) - (3)(-0.00745050593819)}{(16) - (-0.00745050593819)}$$

$$= 2.09430545113$$

Now, $f(2.09430545113) = -0.00274567283813$
Here $f(2.09430545113) = -0.00274567283813$ and $f(2.09388370846) = -0.00745050593819$ have same sign.
Therefore, we replace 2.09388370846 by 2.09430545113 in the current interval $(2.09388370846, 3)$ to obtain new interval of approximation as given below

$$(2.09430545113, 3)$$

$$x_7 = \frac{(2.09430545113)f(3) - (3)f(2.09430545113)}{f(3) - f(2.09430545113)}$$

$$= \frac{(2.09430545113)(16) - (3)(-0.00274567283813)}{(16) - (-0.00274567283813)}$$

$$= 2.09446084577$$

We find that 3 digits immidiately after the decimal point in x_6 and x_7 are same.

Therefore 2.094 is an approximate real root of $x^3 - 2x - 5 = 0$ correct upto 3 decimal places.

14. **Find a real root of $x^3 - 4x - 9 = 0$ by method of False Position correct upto three decimal places**

<u>Solution</u>:
For $f(x) = x^3 - 4x - 9$, we have $f(2) = -9$ and $f(3) = 6$.
Therefore, a root of the equation $x^3 - 4x - 9 = 0$ lies in $(2, 3)$

Now, when an interval $[a, b]$ contains a root of the equatin we use **<u>False Position</u>** method formula $x_i = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ to obtain successive approximations as follows.

$$x_1 = \frac{(2)f(3) - (3)f(2)}{f(3) - f(2)}$$

$$= \frac{(2)(6) - (3)(-9)}{(6) - (-9)}$$

$$= 2.6$$

Now, $f(2.6) = -1.824$
Here $f(2.6) = -1.824$ and $f(2) = -9$ have same sign.
Therefore, we replace 2 by 2.6 in the current interval $(2, 3)$ to obtain new interval of approximation as given below

$$(2.6, 3)$$

$$x_2 = \frac{(2.6)f(3) - (3)f(2.6)}{f(3) - f(2.6)}$$

$$= \frac{(2.6)(6) - (3)(-1.824)}{(6) - (-1.824)}$$

$$= 2.69325153374$$

Now, $f(2.69325153374) = -0.237226510807$
Here $f(2.69325153374) = -0.237226510807$ and $f(2.6) = -1.824$ have same sign.
Therefore, we replace 2.6 by 2.69325153374 in the current interval $(2.6, 3)$ to obtain new interval of approximation as given below

$$(2.69325153374, 3)$$

$$x_3 = \frac{(2.69325153374)f(3) - (3)f(2.69325153374)}{f(3) - f(2.69325153374)}$$

$$= \frac{(2.69325153374)(6) - (3)(-0.237226510807)}{(6) - (-0.237226510807)}$$

$$= 2.70491839693$$

Now, $f(2.70491839693) = -0.0289121838676$
Here $f(2.70491839693) = -0.0289121838676$ and $f(2.69325153374) = -0.237226510807$ have same sign.
Therefore, we replace 2.69325153374 by 2.70491839693 in the current interval $(2.69325153374, 3)$ to obtain new interval of approximation as given below

$$(2.70491839693, 3)$$

$$x_4 = \frac{(2.70491839693)f(3) - (3)f(2.70491839693)}{f(3) - f(2.70491839693)}$$

$$= \frac{(2.70491839693)(6) - (3)(-0.0289121838676)}{(6) - (-0.0289121838676)}$$

$$= 2.70633348696$$

Now, $f(2.70633348696) = -0.00349541816072$
Here $f(2.70633348696) = -0.00349541816072$ and $f(2.70491839693) = -0.0289121838676$ have same sign.
Therefore, we replace 2.70491839693 by 2.70633348696 in the current interval $(2.70491839693, 3)$ to obtain new interval of approximation as given below

$$(2.70633348696, 3)$$

$$x_5 = \frac{(2.70633348696)f(3) - (3)f(2.70633348696)}{f(3) - f(2.70633348696)}$$

$$= \frac{(2.70633348696)(6) - (3)(-0.00349541816072)}{(6) - (-0.00349541816072)}$$

$$= 2.70650446856$$

We find that 3 digits immidiately after the decimal point in x_4 and x_5 are same.

Therefore 2.706 is an approximate real root of $x^3 - 4x - 9 = 0$ correct upto 3 decimal places.

15. State and prove the condition on $\phi(x)$ in Iteration method for convergence of a sequence of approximations.

Answer:
Here, ξ is a real root of an equation $f(x) = 0$. So if we express the equation equivalently as $x = \phi(x)$ then we have

$$\xi = \phi(\xi)$$

Now, let x_0 be the initial approximation chosen in some interval I.
Suppsose, $x_1 = \phi(x_0)$. Therefore we have

$$\xi - x_1 = \phi(\xi) - \phi(x_0)$$

using Langrage's Mean Value theorem the right hand expression can be written as follows.

$$\xi - x_1 = \phi'(\xi_0).(\xi - x_0)\ ; \quad \text{for } \xi_0 \text{ between } x_0 \text{ and } \xi \text{ - - - - } (1)$$

Using $x_{n+1} = \phi(x_n)$ we obtain the sequence $x_1, x_2, x_3, \dots$ and corresponding inequalities given below.

$$\xi - x_2 = \phi'(\xi_1).(\xi - x_1)\ ; \quad \text{for } \xi_1 \text{ between } x_1 \text{ and } \xi \text{ - - - - } (2)$$
$$\xi - x_3 = \phi'(\xi_2).(\xi - x_2)\ ; \quad \text{for } \xi_2 \text{ between } x_2 \text{ and } \xi \text{ - - - - } (3)$$

$$\vdots$$

$$\xi - x_{n+1} = \phi'(\xi_n).(\xi - x_n) \text{ ;} \quad \text{for } \xi_n \text{ between } x_1 \text{ and } \xi \text{ - - - - (n+1)}$$

Now, if we assume that for some number k

$$|\phi'(\xi_i)| \leqslant k < 1, \quad \text{for all } i$$

then above inequalities $(1), (2), \ldots, (n+1)$ give

$$|\xi - x_{i+1}| \leqslant |\xi - x_i| \quad \text{for all i}$$

This implies that if $|\phi'(\xi_i)| < 1$ for all i, then all successive approximations remian in I provided the initial approximation is chosen in I.

Finally, we show that the sequence of successive approximations converges to ξ.

Multiplying the inequalities $(1), (2), \ldots, (n+1)$ and then simplifying we get,

$|\xi - x_{n+1}| \leqslant |\xi - x_0|.|\phi'(\xi_0)|.|\phi'(\xi_1)| \ldots .|\phi'(\xi_n)|$

Since $|\phi'(\xi_i)| \leqslant k$, we get,

$|\xi - x_{n+1}| \leqslant k^{n+1}|\xi - x_0|$

Also since $k < 1$, the right hand side tends to 0 as $n \to \infty$

Therefore the sequence of successive approximations will converge to ξ if $|\phi'(x)| < 1$

If the conditions in the theorem are satisfied then using iteration formula $x_{n+1} = \phi(x_n)$ we can find successive approximations.

Iteration method is linearly convergent.

16. Find a real root of $x^2 + x - 1 = 0$ by iteration method correct upto three decimal places

Solution:
For $f(x) = x^2 + x - 1$, we have $f(0) = -1$ and $f(1) = 1$.
Therefore, a root of the equation $x^2 + x - 1 = 0$ lies in $(0, 1)$

We obtain initial approximation as mid-value in of the interval $x_1 = \dfrac{0+1}{2} = 0.5$

Now let us express given equation $x^2 + x - 1 = 0$ as $x = \dfrac{1}{x+1}$

Let, $\phi(x) = \dfrac{1}{x+1}$

Here, $\phi'(x) = -(x+1)^{-2}$

As the condition $|\phi'(x)| = |-(x+1)^{-2}| < 1, \forall x \in (0, 1)$ is satisfied, the relation

$$x_{n+1} = \phi(x_n)$$

will work for approximations.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{1}{x+1}$$

we get following successive approximations.

$$x_2 = \phi(x_1) = \frac{1}{0.5+1} = 0.666666666667$$
$$x_3 = \phi(x_2) = \frac{1}{0.666666666667+1} = 0.6$$
$$x_4 = \phi(x_3) = \frac{1}{0.6+1} = 0.625$$
$$x_5 = \phi(x_4) = \frac{1}{0.625+1} = 0.615384615385$$
$$x_6 = \phi(x_5) = \frac{1}{0.615384615385+1} = 0.619047619048$$
$$x_7 = \phi(x_6) = \frac{1}{0.619047619048+1} = 0.617647058824$$
$$x_8 = \phi(x_7) = \frac{1}{0.617647058824+1} = 0.618181818182$$
$$x_9 = \phi(x_8) = \frac{1}{0.618181818182+1} = 0.61797752809$$
$$x_{10} = \phi(x_9) = \frac{1}{0.61797752809+1} = 0.618055555556$$
$$x_{11} = \phi(x_{10}) = \frac{1}{0.618055555556+1} = 0.618025751073$$

We find that 3 digits immidiately after the decimal point in x_{10} and x_{11} are same.

Therefore 0.618 is an approximate real root of $x^2 + x - 1 = 0$ correct upto 3 decimal places.

17. Find a real root of $3x = \sin x + 2$ by iteration method correct upto three decimal places

Solution:
For $f(x) = 3x - \sin(x) - 2$, we have $f(0) = -2$ and $f(1) = 0.158529015192$.
Therefore, a root of the equation $3x = \sin x + 2$ lies in $(0, 1)$

We obtain initial approximation as mid-value in of the interval $x_1 = \frac{0+1}{2} = 0.5$

Now let us express given equation $3x = \sin(x) + 2$ as $x = \frac{\sin x + 2}{3}$

Let,

$$\phi(x) = \frac{\sin x + 2}{3}$$

Here, $\phi'(x) = \frac{1}{3}\cos(x)$

As

$$\left|\frac{1}{3}\cos(x)\right| < \frac{1}{3} < 1$$

the condition

$$|\phi'(x)| < 1$$

is satisfied for all $x \in (0, 1)$.

Therefore, the relation

$$x_{n+1} = \phi(x_n)$$

is suitable choice for approximations.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{\sin x + 2}{3}$$

we get following successive approximations.

$$x_2 = \phi(x_1) = \frac{\sin(0.5) + 2}{3} = 0.826475179535$$
$$x_3 = \phi(x_2) = \frac{\sin(0.826475179535) + 2}{3} = 0.911849325329$$
$$x_4 = \phi(x_3) = \frac{\sin(0.911849325329) + 2}{3} = 0.930212468184$$
$$x_5 = \phi(x_4) = \frac{\sin(0.930212468184) + 2}{3} = 0.93391564783$$
$$x_6 = \phi(x_5) = \frac{\sin(0.93391564783) + 2}{3} = 0.934651565636$$
$$x_7 = \phi(x_6) = \frac{\sin(0.934651565636) + 2}{3} = 0.934797374176$$

We find that 3 digits immidiately after the decimal point in x_6 and x_7 are same.

Therefore 0.934 is an approximate real root of $3x = \frac{\sin(x) + 2}{3} = 0$ correct upto 3 decimal places.

> 18. **Find a real root of $\sin x = 10(x - 1)$ by iteration method correct upto three decimal places**

Solution:
For $f(x) = \sin x - 10(x - 1)$, we have $f(1) = 0.841470984808$ and $f(2) = -9.09070257317$.
Therefore, a root of the equation $\sin x - 10(x - 1) = 0$ lies in $(1, 2)$

We obtain initial approximation as mid-value in of the interval $x_1 = \frac{1+2}{2} = 1.5$

Now let us express given equation $\sin x - 10(x-1) = 0$ as $x = \frac{\sin(x)}{10} + 1$

Let, $\phi(x) = \frac{\sin(x)}{10} + 1$

Here, $\phi'(x) = \frac{\cos(x)}{10}$

As

$$\left|\frac{\cos(x)}{10}\right| < \frac{1}{10} < 1$$

the condition

$$|\phi'(x)| < 1$$

is satisfied for all $x \in (1,2)$.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{\sin(x)}{10} + 1$$

we get following successive approximations.

$x_2 = \phi(x_1) = \frac{\sin(1.5)}{10} + 1 = 1.09974949866$

$x_3 = \phi(x_2) = \frac{\sin(1.09974949866)}{10} + 1 = 1.08910937057$

$x_4 = \phi(x_3) = \frac{\sin(1.08910937057)}{10} + 1 = 1.08862146598$

$x_5 = \phi(x_4) = \frac{\sin(1.08862146598)}{10} + 1 = 1.08859885204$

We find that 3 digits immidiately after the decimal point in x_4 and x_5 are same.

Therefore 1.088 is an approximate real root of $\sin x - 10(x-1) = 0$ correct upto 3 decimal places.

19. Find a real root of $\cos x = 3x - 1$ by iteration method correct upto three decimal places

Solution (Only Approximations):

A real root of $3x - \cos x - 1 = 0$ is found in the interval $(0,1)$ and it is

$$x \approx 0.607000$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Iteration Method** by taking

$$\phi(x) = (cos(x) + 1)/3$$

$x_1 = 0.5$

$x_2 = 0.625860853963$
$x_3 = 0.60348637859$
$x_4 = 0.607787348521$
$x_5 = 0.606971188806$
$x_6 = 0.607126454323$
$x_7 = 0.60709693079$

20. Find a real root of $2x = \cos x + 3$ by iteration method correct upto three decimal places

Solution (Only Approximations):
A real root of $2x - cosx - 3 = 0$ is found in the interval $(1, 2)$ and it is

$$x \approx 1.523$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Iteration Method** by taking

$$\phi(x) = (cos(x) + 3)/2$$

$x_1 = 1.5$
$x_2 = 1.53536860083$
$x_3 = 1.5177101577$
$x_4 = 1.52653061928$
$x_5 = 1.52212562642$
$x_6 = 1.52432574358$
$x_7 = 1.52322692968$
$x_8 = 1.52377572938$

$$x \approx 1.523$$

21. Find a real root of $e^x - 3 * x = 0$ by iteration method correct upto three decimal places

Solution (Only Approximations):
A real root of $e^x - 3 * x = 0$ is found in the interval $(0, 1)$ and it is

$$x \approx 0.618$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Iteration Method** by taking

$$\phi(x) = (exp(1)^x)/3$$

$x_1 = 0.5$
$x_2 = 0.5495737569$
$x_3 = 0.577504796052$
$x_4 = 0.59386248532$
$x_5 = 0.603656589392$
$x_6 = 0.609597912327$
$x_7 = 0.61323051092$
$x_8 = 0.615462182139$
$x_9 = 0.616837225129$
$x_{10} = 0.617685986239$
$x_{11} = 0.618210476635$
$x_{12} = 0.618534807139$

$$x \approx 0.618$$

22. First and Second Order Forward Differences

First and Second Order Forward Differences:
Let $x_0, x_1, x_2, \ldots, x_n$ be a set of data. Then for two consecutive values x_i and x_{i+1} the First Order Forward Difference is denoted by Δx_i and it is defined by

$$\boxed{\mathbf{\Delta x_i = x_{i+1} - x_i}}$$

For three consecutive values x_{i-1}, x_i and x_{i+1} the Second Order Forward Difference is denoted by $\Delta^2 x_i$ and it is defined by

$$\boxed{\mathbf{\Delta^2 x_i = \Delta x_i - \Delta x_{i-1}}}$$

23. Discuss the Aitken's Δ^2-Process for approximation of a real root of an equation.

Answer:
Let ξ be a real root of an equation $f(x) = 0$. Let us express the equation equivalently as given below

$$x = \phi(x)$$

As ξ is a root of $f(x) = 0$, it also satisfies $x = \phi(x)$.Therefore,

$$\xi = \phi(\xi)$$

Now, let x_0 be the initial approximation chosen in some interval I.
Suppose, using Iteration methos we have three successive approximations x_{i-1}, x_i and x_{i+1} of ξ. Then for some k such that $|\phi'(x)| < 1|$ we have

$$\xi - x_i \approx k(\xi - x_{i-1}) \quad \text{and} \quad \xi - x_{i+1} \approx k(\xi - x_i)$$

Dividing we can eliminate k and obtain

$$\frac{\xi - x_i}{\xi - x_{i+1}} \approx \frac{\xi - x_{i-1}}{\xi - x_i}$$
$$(\xi - x_i)^2 \approx (\xi - x_{i-1})(\xi - x_{i+1})$$

Simplifying

$$\xi^2 - 2\xi x_i + x_i^2 \approx \xi^2 - \xi(x_{i-1} + x_{i+1}) + x_{i-1}x_{i+1}$$
$$-2\xi x_i + x_i^2 \approx -\xi(x_{i-1} + x_{i+1}) + x_{i-1}x_{i+1}$$
$$\xi(x_{i-1} + x_i + x_{i+1}) \approx x_{i-1}x_{i+1} - x_i^2$$

On the right hand side adding and subtracting $x_{i+1}^2 - 2x_i x_{i+1}$

$$\xi(x_{i+1} - 2x_i + x_{i-1}) \approx x_{i+1}^2 - 2x_i x_{i+1} + x_{i-1}x_{i+1} - x_i^2 - x_{i+1}^2 + 2x_i x_{i+1}$$
$$\xi(x_{i+1} - 2x_i + x_{i-1}) \approx x_{i+1}(x_{i+1} - 2x_i + x_{i-1}) - (x_{i+1} - x_i)^2$$
$$\xi \approx x_{i+1} - \frac{(x_{i+1} - x_i)^2}{x_{i+1} - 2x_i + x_{i-1}}$$

Using forward difference Δ notations we get,

$$\Delta x_i = x_{i+1} - x_i \quad \text{and} \quad \Delta^2 x_{i-1} = x_{i+1} - 2x_i + x_{i-1}$$

Therefore above formual can be expressed as

$$\xi \approx x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$$

As the it gives next appriximation to ξ we have the Aitken's Δ^2 process approximation formula as given below

$$\mathbf{x_{i+2} = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}}$$

24. $x = \dfrac{1}{(x+1)^2}$

Solution:

For $f(x) = x(x+4)^2 - 1$, we have $f(0) = -1$ and $f(1) = 24$.
Therefore, a root of the equation $x(x+4)^2 - 1 = 0$ lies in $(0, 1)$

We obtain initial approximation as mid-value in of the interval $x_1 = \dfrac{0+1}{2} = 0.5$

Now let us express given equation $x(x+4)^2 - 1 = 0$ as $x = \dfrac{1}{(x+4)^2}$

Let, $\phi(x) = \dfrac{1}{(x+4)^2}$

Here, $\phi'(x) = -2\ (x+4)^{-3}$

As the condition $|\phi'(x)| = |-2\ (x+4)^{-3}| < 1, \forall x \in (0,1)$ is satisfied, the relation

$$x_{n+1} = \phi(x_n)$$

will work for approximations.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{1}{(x+4)^2}$$

we get following successive approximations.

$$x_2 = \phi(x_1) = \frac{1}{(0.0493827160494 + 4)^2} = 0.0493827160494$$
$$x_3 = \phi(x_2) = \frac{1}{(0.0609849048186 + 4)^2} = 0.0609849048186$$

Now, we shall use the formula $x_{i+2} = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ for **Aitken's Δ^2 Process** to obtain successive approximations as follows.

We have $\Delta x_1 = 0.0493827160494 - 0.5 = -0.450617283951$

Also $\Delta x_2 = 0.0609849048186 - 0.0493827160494 = 0.0116021887692$

$\Delta^2 x_1 = 0.0116021887692 - -0.450617283951 = 0.46221947272$

Therefore,

$$\begin{aligned} x_4 &= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} \\ &= 0.0609849048186 - \frac{(0.0116021887692)^2}{(0.46221947272)} \\ &= 0.0606936778319 \end{aligned}$$

We find that 3 digits immidiately after the decimal point in x_3 and x_4 are same.

Therefore 0.060 is an approximate real root of $x(x+4)^2 - 1 = 0$ correct upto 3 decimal places.

25. Solve $x^3 + x^2 - 1 = 0$ by Aitken's Δ^2 process correct upto three decimal places

Solution:
For $f(x) = x^3 + x^2 - 1$, we have $f(0) = -1$ and $f(1) = 1$.
Therefore, a root of the equation $x^3 + x^2 - 1 = 0$ lies in $(0,1)$

We obtain initial approximation as mid-value in of the interval $x_1 = \dfrac{0+1}{2} = 0.5$

Now let us express given equation $x^3 + x^2 - 1 = 0$ as $x = \dfrac{1}{\sqrt{x+1}}$

Let, $\phi(x) = \dfrac{1}{\sqrt{x+1}}$

Here, $\phi'(x) = -\frac{1}{2\,(x+1)^{\frac{3}{2}}}$

As the condition $|\phi'(x)| = |-\frac{1}{2\,(x+1)^{\frac{3}{2}}}| < 1, \forall x \in (0,1)$ is satisfied, the relation

$$x_{n+1} = \phi(x_n)$$

will work for approximations.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{1}{\sqrt{x+1}}$$

we get following successive approximations.

$$x_2 = \phi(x_1) = \frac{1}{\sqrt{0.816496580928+1}} = 0.816496580928$$
$$x_3 = \phi(x_2) = \frac{1}{\sqrt{0.741963784303+1}} = 0.741963784303$$

Now, we shall use the formula $x_{i+2} = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ for **Aitken's Δ^2 Process** to obtain successive approximations as follows.

We have $\Delta x_1 = 0.816496580928 - 0.5 = 0.316496580928$

Also $\Delta x_2 = 0.741963784303 - 0.816496580928 = -0.074532796625$

$\Delta^2 x_1 = -0.074532796625 - 0.316496580928 = -0.391029377553$

Therefore,

$$\begin{aligned} x_4 &= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} \\ &= 0.741963784303 - \frac{(-0.074532796625)^2}{(-0.391029377553)} \\ &= 0.756170230395 \end{aligned}$$

Also $\Delta x_3 = 0.756170230395 - 0.741963784303 = 0.0142064460924$

$\Delta^2 x_2 = 0.0142064460924 - -0.074532796625 = 0.0887392427174$

Therefore,

$$\begin{aligned} x_5 &= x_4 - \frac{(\Delta x_3)^2}{\Delta^2 x_2} \\ &= 0.756170230395 - \frac{(0.0142064460924)^2}{(0.0887392427174)} \\ &= 0.753895891507 \end{aligned}$$

Also $\Delta x_4 = 0.753895891507 - 0.756170230395 = -0.00227433888769$

$\Delta^2 x_3 = -0.00227433888769 - 0.0142064460924 = -0.0164807849801$

Therefore,

$$x_6 = x_5 - \frac{(\Delta x_4)^2}{\Delta^2 x_3}$$
$$= 0.753895891507 - \frac{(-0.00227433888769)^2}{(-0.0164807849801)}$$
$$= 0.754209748971$$

Also $\Delta x_5 = 0.754209748971 - 0.753895891507 = 0.000313857463846$

$\Delta^2 x_4 = 0.000313857463846 - -0.00227433888769 = 0.00258819635153$

Therefore,

$$x_7 = x_6 - \frac{(\Delta x_5)^2}{\Delta^2 x_4}$$
$$= 0.754209748971 - \frac{(0.000313857463846)^2}{(0.00258819635153)}$$
$$= 0.754171689066$$

We find that 3 digits immidiately after the decimal point in x_6 and x_7 are same.

Therefore 0.754 is an approximate real root of $x^3 + x^2 - 1 = 0$ correct upto 3 decimal places.

26. Solve $2x = \cos x + 3$ by Aitken's Δ^2 process correct upto three decimal places

Solution:
For $f(x) = 2x - \cos x + 3$, we have $f(1) = -1.54030230587$ and $f(2) = 1.41614683655$.
Therefore, a root of the equation $2x - \cos x + 3 = 0$ lies in $(1, 2)$

We obtain initial approximation as mid-value in of the interval $x_1 = \frac{1+2}{2} = 1.5$

Now let us express given equation $2x - \cos x + 3 = 0$ as $x = \frac{\cos(x) + 3}{2}$

Let, $\phi(x) = \frac{\cos(x) + 3}{2}$

Here, $\phi'(x) = -\frac{\sin(x)}{2}$

As the condition $|\phi'(x)| = |-\frac{\sin(x)}{2}| < 1, \forall x \in (1, 2)$ is satisfied, the relation

$$x_{n+1} = \phi(x_n)$$

will work for approximations.

Using the **Iteration Method** by taking

$$\phi(x) = \frac{\cos(x) + 3}{2}$$

we get following successive approximations.

$$x_2 = \phi(x_1) = \frac{\cos(1.53536860083) + 3}{2} = 1.53536860083$$

$$x_3 = \phi(x_2) = \frac{\cos(1.5177101577) + 3}{2} = 1.5177101577$$

Now, we shall use the formula $x_{i+2} = x_{i+1} - \dfrac{(\Delta x_i)^2}{\Delta^2 x_{i-1}}$ for **Aitken's Δ^2 Process** to obtain successive approximations as follows.

We have $\Delta x_1 = 1.53536860083 - 1.5 = 0.0353686008339$

Also $\Delta x_2 = 1.5177101577 - 1.53536860083 = -0.0176584431359$

$\Delta^2 x_1 = -0.0176584431359 - 0.0353686008339 = -0.0530270439697$

Therefore,

$$\begin{aligned} x_4 &= x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} \\ &= 1.5177101577 - \frac{(-0.0176584431359)^2}{(-0.0530270439697)} \\ &= 1.52359056495 \end{aligned}$$

Also $\Delta x_3 = 1.52359056495 - 1.5177101577 = 0.00588040725334$

$\Delta^2 x_2 = 0.00588040725334 - -0.0176584431359 = 0.0235388503892$

Therefore,

$$\begin{aligned} x_5 &= x_4 - \frac{(\Delta x_3)^2}{\Delta^2 x_2} \\ &= 1.52359056495 - \frac{(0.00588040725334)^2}{(0.0235388503892)} \\ &= 1.52212153869 \end{aligned}$$

Also $\Delta x_4 = 1.52212153869 - 1.52359056495 = -0.00146902626481$

$\Delta^2 x_3 = -0.00146902626481 - 0.00588040725334 = -0.00734943351814$

Therefore,

$$\begin{aligned} x_6 &= x_5 - \frac{(\Delta x_4)^2}{\Delta^2 x_3} \\ &= 1.52212153869 - \frac{(-0.00146902626481)^2}{(-0.00734943351814)} \\ &= 1.52241517195 \end{aligned}$$

We find that 3 digits immidiately after the decimal point in x_5 and x_6 are same.

Therefore 1.522 is an approximate real root of $2x - \cos x + 3 = 0$ correct upto 3 decimal places.

27. Solve $x - \sin x = \dfrac{1}{2}$ by Aitken's Δ^2 process correct upto four decimal places

Solution (Only approximations):
A real root of $x - sin(x) = \frac{1}{2}$ is found in the interval $(1, 2)$ and it is

$$x \approx 1.4973$$

, correct upto 4 decimal places.

Following are successive approximations obtained using **Aitken's Δ^2 Process**

$x_1 = 1.5$
$x_2 = 1.4974949866$
$x_3 = 1.49731465947$
$x_4 = 1.4973006714$

$$x \approx 1.4973$$

28. Describe the Newton-Raphson method for approximation of real root of an equation.

Answer:
Let $f(x) = 0$ be an equation and x_0 be initial approximation to a real root, say ξ, of the equation.
Suppose $\xi = x_0 + h$ for some h

As ξ is a root of the equation, we have $f(\xi) = 0$

Therefore $f(x_0 + h) = 0$

If f satisfies all the conditions for Taylor's series expansion then

$$f(x_0) + h.f'(x_0) + \frac{h^2}{2!}f''(x_0) + ... = 0$$

Neglecting the terms with second and higher order derivatives we have

$$f(x_0) + h.f'(x_0) \approx 0$$

Therefore,

$$h \approx -\frac{f(x_0)}{f'(x_0)}$$

Substituting in $\xi = x_0 + h$ we get

$$\xi \approx x_0 - \frac{f(x_0)}{f'(x_0)}$$

Using similar arguments we can obtain a generalized formula called Newton-Raphson appoximation formula as

$$x_{n+1} \approx x_n - \frac{f(x_n)}{f'(x_n)}$$

29. Find a real root of $x^3 - 3 * x + 7 = 0$, correct upto four decimal places, by Newton-Raphson method

Solution:
For the function $f(x) = x^3 - 3x + 7$, we have its detivative $f'(x) = 3\,x^2 - 3$
Now, For $f(x) = x^3 - 3x + 7$, we have $f(-3) = -11$ and $f(-2) = 5$.
Therefore, a root of the equation $x^3 - 3x + 7 = 0$ lies in $(-3, -2)$

Let $X_1 = \dfrac{-3 + -2}{2} = -2.5$
Now, starting with initial approximation x_1 and using the approximation formula of **Newton-Raphson** method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_i)}$$

we obtain successive approximations as follows.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -2.5 - \frac{f(-2.5)}{f'(-2.5)}$$

$$= -2.5 - \frac{-2.5^3 - 3 - 2.5 + 7}{3\,-2.5^2 - 3}$$

$$= -2.5 - \frac{-1.125}{15.75}$$

$$= -2.42857142857$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -2.42857142857 - \frac{f(-2.42857142857)}{f'(-2.42857142857)}$$

$$= -2.42857142857 - \frac{-2.42857142857^3 - 3 - 2.42857142857 + 7}{3\,-2.42857142857^2 - 3}$$

$$= -2.42857142857 - \frac{-0.0379008746356}{14.693877551}$$

$$= -2.42599206349$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= -2.42599206349 - \frac{f(-2.42599206349)}{f'(-2.42599206349)}$$

$$= -2.42599206349 - \frac{-2.42599206349^3 - 3 - 2.42599206349 + 7}{3\,-2.42599206349^2 - 3}$$

$$= -2.42599206349 - \frac{-4.84556012861e - 005}{14.6563124764}$$

$$= -2.42598875737$$

We find that 4 digits immediately after the decimal point in x_3 and x_4 are same.

Therefore -2.4259 is an approximate real root of $x^3 - 3x + 7 = 0$ correct upto 4 decimal places.

30. **Find a real root of $x \sin x + \cos x = 0$, correct upto three decimal places, by Newton-Raphson method**

Solution:
For the function $f(x) = x\sin(x) + \cos(x)$, we have its detivative $f'(x) = x\ \cos(x)$
Now, For $f(x) = x\sin(x)+\cos(x)$, we have $f(2) = 1.4024480171$ and $f(3) = -0.566632472421$.
Therefore, a root of the equation $x\sin(x) + \cos(x) = 0$ lies in $(2, 3)$

Let $X_1 = \frac{2+3}{2} = 2.5$
Now, starting with initial approximation x_1 and using the approximation formula of **Newton-Raphson**

method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_i)}$$

we obtain successive approximations as follows.

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= 2.5 - \frac{f(2.5)}{f'(2.5)} \\ &= 2.5 - \frac{2.5\sin(2.5) + \cos(2.5)}{2.5\ \cos(2.5)} \\ &= 2.5 - \frac{0.695036744713}{-2.00285903887} \\ &= 2.84702229724 \end{aligned}$$

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= 2.84702229724 - \frac{f(2.84702229724)}{f'(2.84702229724)} \\ &= 2.84702229724 - \frac{2.84702229724\sin(2.84702229724) + \cos(2.84702229724)}{2.84702229724\ \cos(2.84702229724)} \\ &= 2.84702229724 - \frac{-0.130354574112}{-2.72439241626} \\ &= 2.79917508795 \end{aligned}$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.79917508795 - \frac{f(2.79917508795)}{f'(2.79917508795)}$$

$$= 2.79917508795 - \frac{2.79917508795\sin(2.79917508795) + \cos(2.79917508795)}{2.79917508795\ \cos(2.79917508795)}$$

$$= 2.79917508795 - \frac{-0.00207985856872}{-2.63667089386}$$

$$= 2.79838626802$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= 2.79838626802 - \frac{f(2.79838626802)}{f'(2.79838626802)}$$

$$= 2.79838626802 - \frac{2.79838626802\sin(2.79838626802) + \cos(2.79838626802)}{2.79838626802\ \cos(2.79838626802)}$$

$$= 2.79838626802 - \frac{-5.85626972138e - 007}{-2.63518587234}$$

$$= 2.79838604578$$

We find that 3 digits immediately after the decimal point in x_4 and x_5 are same.

Therefore 2.798 is an approximate real root of $x\sin(x) + \cos(x) = 0$ correct upto 3 decimal places.

31. Find a real root of $x^3 - 3x + 5 = 0$, correct upto three decimal places, by Newton-Raphson method

Solution (Only Approximations):
A real root of $x^3 - 3x + 5 = 0$ is found in the interval $(-3, -2)$ and it is

$$x \approx -2.279$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Newton-Raphson Method**

$x_1 = -2.5$

$x_2 = -2.30158730159$
$x_3 = -2.27929069011$
$x_4 = -2.27901882633$

32. Find a real root of $x^3 - 2x - 5 = 0$, correct upto three decimal places, by Newton-Raphson method

Solution (Only Approximations):
A real root of $x^3 - 2x - 5 = 0$ is found in the interval $(2, 3)$ and it is

$$x \approx 2.094$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Newton-Raphson Method**

$x_1 = 2.5$
$x_2 = 2.16417910448$
$x_3 = 2.09713535581$
$x_4 = 2.09455523239$
$x_5 = 2.09455148155$

33. Find a real root of $x^3 - 5x + 3 = 0$, correct upto three decimal places, by Newton-Raphson method

Solution (Only Approximations):
A real root of $x^3 - 5x + 3 = 0$ is found in the interval $(0, 1)$ and it is

$$x \approx 0.656$$

, correct upto 3 decimal places.

Following are successive approximations obtained using **Newton-Raphson Method**

$x_1 = 0.5$
$x_2 = 0.647058823529$
$x_3 = 0.656572795477$
$x_4 = 0.656620429841$

34. Using the Newton-Raphson method, establish the iterative formula

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

to calulate the cube root of N

Answer:

If x_n is an approximation to the root of an equation $f(x) = 0$, where f is differentiable on an interval containing x_n, then by Newton-Raphson approximation formula the next approximation is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Now suppose x is a real cube root of N.
Then we have, $x^3 = N$
Therefore, $x^3 - N = 0$

Now, to approximate the cube root , let us define

$$f(x) = x^3 - N$$

Here,

$$f'(x) = 3x^2$$

Therefore, if x_n is an approximation to x then using Newton-Raphson approximation formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

we get the next approximation by

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$= \frac{1}{3}\left[\frac{3x_n^3 - x_n^3 + N}{x_n^2}\right]$$

$$= \frac{1}{3}\left[\frac{2x_n^3 + N}{x_n^2}\right]$$

$$= \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

Thus the iterative formula for finding successive approximations of cube root of a number N is

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$$

35. Using the formula $x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$,calulate the cube root of 10 correct upto four decimal places.

Cube root of 17 is found in the interval $(2,3)$ and it is

$$x \approx 2.5712$$

, correct upto 4 decimal places.

Following are successive approximations obtained using the iterative formula

$x_1 = 2.5$
$x_2 = 2.57333333333$
$x_3 = 2.5712832261$
$x_4 = 2.57128159066$

$$x \approx 2.5712$$

Solution:
We want to find out cube root of N=17
Since $8 < 17 < 27$, cube root of 17 lies between 2 and 3
i.e. cube root of 17 lies in $(2,3)$

So we can take the mid-value $x_1 = \dfrac{2+3}{2} = 2.5$ as initial approximation
Now, we have the iterative approximation formula

$$x_{n+1} = \frac{1}{3}\left[2(x_n) + \frac{N}{(x_n)^2}\right]$$

we obtain successive approximations as follows.

$x_2 = \dfrac{1}{3}\left[2(2.5) + \dfrac{17}{(2.5)^2}\right] = 2.57333333333$
$x_3 = \dfrac{1}{3}\left[2(2.57333333333) + \dfrac{17}{(2.57333333333)^2}\right] = 2.5712832261$
$x_4 = \dfrac{1}{3}\left[2(2.5712832261) + \dfrac{17}{(2.5712832261)^2}\right] = 2.57128159066$

We find that 4 digits immediately after the decimal point in x_3 and x_4 are same.

Therefore 2.5712 is an approximate cube root of 17 correct upto 4 decimal places.

36. **Using the formula $x_{n+1} = \dfrac{1}{3}\left[2x_n + \dfrac{N}{x_n^2}\right]$,calulate the cube root of 12 correct upto three decimal places.**

Solution (Only Approximations):
Cube root of 12 is found in the interval $(2,3)$ and it is

$$x \approx 2.289$$

, correct upto 3 decimal places.

Following are successive approximations obtained using the iterative formula

$x_1 = 2.5$
$x_2 = 2.30666666667$
$x_3 = 2.28955698858$
$x_4 = 2.28942849232$

37. **Using the formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$,calulate the square root of 27 correct upto three decimal places.**

Solution:

We want to find out square root of N=27
Since $25 < 27 < 36$, square root of 27 lies between 5 and 6
i.e. square root of 27 lies in $(5, 6)$

So we can take the mid-value $x_1 = \frac{5+6}{2} = 5.5$ as initial approximation
Now, we have the iterative approximation formula

$$x_{n+1} = \frac{1}{2}\left[(x_n) + \frac{N}{(x_n)}\right]$$

we obtain successive approximations as follows.

$$x_2 = \frac{1}{2}\left[(5.5) + \frac{27}{(5.5)}\right] = 5.20454545455$$
$$x_3 = \frac{1}{2}\left[(5.20454545455) + \frac{27}{(5.20454545455)}\right] = 5.19615919015$$
$$x_4 = \frac{1}{2}\left[(5.19615919015) + \frac{27}{(5.19615919015)}\right] = 5.19615242271$$

We find that 3 digits immediately after the decimal point in x_3 and x_4 are same.

Therefore 5.196 is an approximate square root of 27 correct upto 3 decimal places.

38. **Using the formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$,calulate the square root of 8 correct upto three decimal places.**

Solution (Only Approximations):

Square root of 8 is found in the interval $(2, 3)$ and it is

$$x \approx 2.828$$

, correct upto 3 decimal places.

Following are successive approximations obtained using the iterative formula

$x_1 = 2.5$
$x_2 = 2.85$

$x_3 = 2.82850877193$
$x_4 = 2.82842712592$

> **39. Using the formula $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$,calulate the square root of 5 correct upto three decimal places.**

Solution (Only Approximations):

Square root of 5 is found in the interval $(2, 3)$ and it is

$$x \approx 2.236$$

, correct upto 3 decimal places.

Following are successive approximations obtained using the iterative formula

$x_1 = 2.5$
$x_2 = 2.25$
$x_3 = 2.23611111111$
$x_4 = 2.23606797792$

> **40. Describe Ramanujan's method for finding smallest root of an eauation.**

Assume that an equation to be solved is expressed in the following form.

$$f(x) = 1 - \left(a_1 x + a_2 x^2 + a_3 x^3 + \dots\right) \text{ - - - (1)}$$

To find the **smallest** root of this equation using Ramanujan's method we use following steps to find constants b_i and the ratios $\frac{b_{i-1}}{b_i}$, known as **convergents**.

(1) Take $b_1 = 1$
(2) Find $b_2 = a_1 b_1$ (As $b_1 = 1$ we have $b_2 = a_1$)
(3) Evaluate $\frac{b_1}{b_2}$
(4) Find $b_3 = a_1 b_2 + a_2 b_1$
(5) Evaluate $\frac{b_2}{b_3}$
(6) Find $b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1$
(7) Evaluate $\frac{b_3}{b_4}$
(8) continue similarly by taking

$$b_k = a_1 b_{k-1} + a_2 b_{k-2} + a_3 b_{k-3} + \dots + +a_{k-1} b_1$$

then evaluating

$$\frac{b_{k-1}}{b_k}$$

The convergents $\frac{b_{i-1}}{b_i}$ are the approximations to the smallest root of the equation (1). We stop the proceess when two consecutive convergents match upto desired acuracy and get the approximate smallest root.

41. Using Ramanujan's method find the smallest root of $x^3-9x^2+26x-24=0$

Given equation is

$$x^3 - 9x^2 + 26x - 24 = 0$$

Therefore,

$$-\frac{1}{24}x^3 + \frac{3}{8}x^2 - \frac{13}{12}x + 1 = 0$$

Therefore,

$$1 - \frac{13}{12}x + \frac{3}{8}x^2 - \frac{1}{24}x^3 = 0$$

Therefore,

$$1 - \left(\frac{13}{12}x - \frac{3}{8}x^2 + \frac{1}{24}x^3\right) = 0$$

Comparing the L.H.S. of the equation with

$$1 - (a_1x + a_2x^2 + a_3x^3 + \dots)$$

We get the following,

$$a_1 = 13/12,\ a_2 = -3/8,\ a_3 = 1/24,\ a_4 = a_5 = a_6 \cdots = 0$$

We determine the constants b_i and the convergents $\frac{b_{i-1}}{b_i}$ as given below.
First take $b_1 = 1$

Then, for each sussessive b_k we shall use the following formula,

$$b_k = a_1b_{k-1} + a_2b_{k-2} + a_3b_{k-3} + \cdots + +a_{k-1}b_1$$

Using the formula we get,

$$b_2 = a_1b_1 = a_1 = 1.08333333$$

Therefore,

$$\frac{b_1}{b_2} = \frac{1}{13/12} = 0.92307692$$

Now,

$$b_3 = a_1b_2 + a_2b_1$$

$$\therefore\ b_3 = (13/12)(1.08333333) + (-3/8)(1) = 0.79861111$$

Therefore,

$$\frac{b_2}{b_3} = \frac{1.08333333}{0.79861111} = 1.35652174$$

Now,

$$b_4 = a_1b_3 + a_2b_2 + a_3b_1$$

$$\therefore\ b_4 = (13/12)(0.79861111) + (-3/8)(1.08333333) + (1/24)(1) = 0.5005787$$

Therefore,

$$\frac{b_3}{b_4} = \frac{0.79861111}{0.5005787} = 1.59537572$$

Now,

$$b_5 = a_1b_4 + a_2b_3 + a_3b_2$$

$$\therefore\ b_5 = (13/12)(0.5005787) + (-3/8)(0.79861111) + (1/24)(1.08333333) = 0.28795332$$

Therefore,

$$\frac{b_4}{b_5} = \frac{0.5005787}{0.28795332} = 1.73840228$$

Now,

$$b_6 = a_1b_5 + a_2b_4 + a_3b_3$$

$$\therefore\ b_6 = (13/12)(0.28795332) + (-3/8)(0.5005787) + (1/24)(0.79861111) = 0.15750788$$

Therefore,

$$\frac{b_5}{b_6} = \frac{0.28795332}{0.15750788} = 1.8281836$$

Now,

$$b_7 = a_1b_6 + a_2b_5 + a_3b_4$$

$$\therefore\ b_7 = (13/12)(0.15750788) + (-3/8)(0.28795332) + (1/24)(0.5005787) = 0.08350848$$

Therefore,

$$\frac{b_6}{b_7} = \frac{0.15750788}{0.08350848} = 1.88613022$$

Now,

$$b_8 = a_1b_7 + a_2b_6 + a_3b_5$$

$$\therefore\ b_8 = (13/12)(0.08350848) + (-3/8)(0.15750788) + (1/24)(0.28795332) = 0.04340013$$

Therefore,

$$\frac{b_7}{b_8} = \frac{0.08350848}{0.04340013} = 1.92415303$$

Now,

$$b_9 = a_1b_8 + a_2b_7 + a_3b_6$$

$$\therefore\ b_9 = (13/12)(0.04340013) + (-3/8)(0.08350848) + (1/24)(0.15750788) = 0.02226395$$

Therefore,

$$\frac{b_8}{b_9} = \frac{0.04340013}{0.02226395} = 1.94934531$$

Now,

$$b_{10} = a_1b_9 + a_2b_8 + a_3b_7$$

$$\therefore\ b_{10} = (13/12)(0.02226395) + (-3/8)(0.04340013) + (1/24)(0.08350848) = 0.01132375$$

Therefore,

$$\frac{b_9}{b_{10}} = \frac{0.02226395}{0.01132375} = 1.96612837$$

Now,

$$b_{11} = a_1b_{10} + a_2b_9 + a_3b_8$$

$$\therefore\ b_{11} = (13/12)(0.01132375) + (-3/8)(0.02226395) + (1/24)(0.04340013) = 0.00572676$$

Therefore,

$$\frac{b_{10}}{b_{11}} = \frac{0.01132375}{0.00572676} = 1.97734168$$

Now,

$$b_{12} = a_1b_{11} + a_2b_{10} + a_3b_9$$

$$\therefore\ b_{12} = (13/12)(0.00572676) + (-3/8)(0.01132375) + (1/24)(0.02226395) = 0.00288524$$

Therefore,

$$\frac{b_{11}}{b_{12}} = \frac{0.00572676}{0.00288524} = 1.98484366$$

Now,

$$b_{13} = a_1b_{12} + a_2b_{11} + a_3b_{10}$$

$$\therefore\ b_{13} = (13/12)(0.00288524) + (-3/8)(0.00572676) + (1/24)(0.01132375) = 0.00144997$$

Therefore,

$$\frac{b_{12}}{b_{13}} = \frac{0.00288524}{0.00144997} = 1.98986476$$

Now,

$$b_{14} = a_1 b_{13} + a_2 b_{12} + a_3 b_{11}$$

$$\therefore\ b_{14} = (13/12)(0.00144997) + (-3/8)(0.00288524) + (1/24)(0.00572676) = 0.00072745$$

Therefore,

$$\frac{b_{13}}{b_{14}} = \frac{0.00144997}{0.00072745} = 1.99322509$$

Now,

$$b_{15} = a_1 b_{14} + a_2 b_{13} + a_3 b_{12}$$

$$\therefore\ b_{15} = (13/12)(0.00072745) + (-3/8)(0.00144997) + (1/24)(0.00288524) = 0.00036455$$

Therefore,

$$\frac{b_{14}}{b_{15}} = \frac{0.00072745}{0.00036455} = 1.99547318$$

Now,

$$b_{16} = a_1 b_{15} + a_2 b_{14} + a_3 b_{13}$$

$$\therefore\ b_{16} = (13/12)(0.00036455) + (-3/8)(0.00072745) + (1/24)(0.00144997) = 0.00018255$$

Therefore,

$$\frac{b_{15}}{b_{16}} = \frac{0.00036455}{0.00018255} = 1.99697648$$

Now,

$$b_{17} = a_1 b_{16} + a_2 b_{15} + a_3 b_{14}$$

$$\therefore\ b_{17} = (13/12)(0.00018255) + (-3/8)(0.00036455) + (1/24)(0.00072745) = 0.00009137$$

Therefore,

$$\frac{b_{16}}{b_{17}} = \frac{0.00018255}{0.00009137} = 1.99798126$$

Now,

$$b_{18} = a_1 b_{17} + a_2 b_{16} + a_3 b_{15}$$

$$\therefore\ b_{18} = (13/12)(0.00009137) + (-3/8)(0.00018255) + (1/24)(0.00036455) = 0.00004571$$

Therefore,

$$\frac{b_{17}}{b_{18}} = \frac{0.00009137}{0.00004571} = 1.99865253$$

Now,

$$b_{19} = a_1 b_{18} + a_2 b_{17} + a_3 b_{16}$$

$$\therefore\ b_{19} = (13/12)(0.00004571) + (-3/8)(0.00009137) + (1/24)(0.00018255) = 0.00002287$$

Therefore,

$$\frac{b_{18}}{b_{19}} = \frac{0.00004571}{0.00002287} = 1.99910082$$

Now,

$$b_{20} = a_1 b_{19} + a_2 b_{18} + a_3 b_{17}$$

$$\therefore\ b_{20} = (13/12)(0.00002287) + (-3/8)(0.00004571) + (1/24)(0.00009137) = 0.00001144$$

Therefore,

$$\frac{b_{19}}{b_{20}} = \frac{0.00002287}{0.00001144} = 1.99940009$$

Hence, the smallest root of the equation correct upto 3 decimal places is

$$x \approx 1.999$$

42. **Using Ramanujan's method find the smallest root of $xe^x = 1$ correct upto 3 decimal places.**

We know that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$$

Therefore, given eaquation can be witten as

$$x(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots) = 1$$

Therefore,

$$1 - \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \frac{x^6}{5!} \ldots\right) = 0$$

Comparing the L.H.S. of the equation with

$$1 - \left(a_1 x + a_2 x^2 + a_3 x^3 + \ldots\right)$$

We get the following,

$$a_1 = 1, a_2 = 1, a_3 = 1/2, a_4 = 1/6, a_5 = 1/24, a_6 = 1/120, a_7 = 1/720 \ldots$$

We determine the constants b_i and the convergents $\frac{b_{i-1}}{b_i}$ as given below.
First take $b_1 = 1$

Then, for each sussessive b_k we shall use the following formula,

$$b_i = a_1 b_{i-1} + a_2 b_{i-2} + a_3 b_{i-3} + \cdots + +a_{i-1} b_1$$

Therefore,

$$b_2 = a_1 b_1 = a_1 = 1$$

Therefore,

$$\frac{b_1}{b_2} = \frac{1}{1} = 1$$

Now,

$$b_3 = a_1 b_2 + a_2 b_1$$
$$\therefore\ b_3 = (1)(1) + (1)(1) = 2$$

Therefore,

$$\frac{b_2}{b_3} = \frac{1}{2} = 0.5$$

Now,

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1$$
$$\therefore\ b_4 = (1)(2) + (1)(1) + (1/2)(1) = 3.5$$

Therefore,

$$\frac{b_3}{b_4} = \frac{2}{3.5} = 0.57142857$$

Now,

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$$
$$\therefore\ b_5 = (1)(3.5) + (1)(2) + (1/2)(1) + (1/6)(1) = 6.16666667$$

Therefore,

$$\frac{b_4}{b_5} = \frac{3.5}{6.16666667} = 0.56756757$$

Now,

$$b_6 = a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2 + a_5b_1$$

$$\therefore\ b_6 = (1)(6.16666667) + (1)(3.5) + (1/2)(2) + (1/6)(1) + (1/24)(1) = 10.875$$

Therefore,

$$\frac{b_5}{b_6} = \frac{6.16666667}{10.875} = 0.56704981$$

Hence, the smallest root of the equation correct upto 3 decimal places is

$$x \approx 0.5670$$

43. Using Ramanujan's method find the smallest root of $3x = cox + 1$

We have infinite series for $\cos x$ given by,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

Substituting in the equation, we get,

$$3x = \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right] + 1$$

Therefore,

$$2 - 3x - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \dots = 0$$

Dividing with the lowest degree term 2 we get,

$$1 - \frac{3}{2}x - \frac{1}{4}x^2 + \frac{1}{48}x^4 - \frac{1}{1440}x^6 + \frac{1}{80640}x^8 - \dots = 0$$

$$\therefore\ 1 - \left[\frac{3}{2}x + \frac{1}{4}x^2 - \frac{1}{48}x^4 + \frac{1}{1440}x^6 - \frac{1}{80640}x^8 + \dots\right] = 0$$

Comparing the R.H.S. of the equation with

$$1 - \left(a_1x + a_2x^2 + a_3x^3 + \dots\right)$$

We get the following,

$$a_1 = 3/2, a_2 = 1/4, a_3 = 0, a_4 = -1/48, a_5 = 0, a_6 = 1/1440, a_7 = 0, a_8 = -1/80640, \dots$$

We determine the constants b_i and the convergents $\frac{b_{i-1}}{b_i}$ as given below.

First take $b_1 = 1$

Then, for each sussessive b_k we shall use the following formula,

$$b_i = a_1b_{i-1} + a_2b_{i-2} + a_3b_{i-3} + \dots + +a_{i-1}b_1$$

Therefore,

$$b_2 = a_1b_1 = a_1 = 1.5$$

Therefore,

$$\frac{b_1}{b_2} = \frac{1}{3/2} = 0.66666667$$

Now,

$$b_3 = a_1b_2 + a_2b_1$$

$$\therefore\ b_3 = (3/2)(1.5) + (1/4)(1) = 2.5$$

Therefore,

$$\frac{b_2}{b_3} = \frac{1.5}{2.5} = 0.6$$

Now,

$$b_4 = a_1b_3 + a_2b_2 + a_3b_1$$

$$\therefore\ b_4 = (3/2)(2.5) + (1/4)(1.5) + (0)(1) = 4.125$$

Therefore,

$$\frac{b_3}{b_4} = \frac{2.5}{4.125} = 0.60606061$$

Now,

$$b_5 = a_1b_4 + a_2b_3 + a_3b_2 + a_4b_1$$

$$\therefore\ b_5 = (3/2)(4.125) + (1/4)(2.5) + (0)(1.5) + (-1/48)(1) = 6.79166667$$

Therefore,

$$\frac{b_4}{b_5} = \frac{4.125}{6.79166667} = 0.60736196$$

Now,

$$b_6 = a_1b_5 + a_2b_4 + a_3b_3 + a_4b_2 + a_5b_1$$

$$\therefore\ b_6 = (3/2)(6.79166667) + (1/4)(4.125) + (0)(2.5) + (-1/48)(1.5) + (0)(1) = 11.1875$$

Therefore,

$$\frac{b_5}{b_6} = \frac{6.79166667}{11.1875} = 0.60707635$$

Now,

$$b_7 = a_1b_6 + a_2b_5 + a_3b_4 + a_4b_3 + a_5b_2 + a_6b_1$$

$$\therefore\ b_7 = (3/2)(11.1875)+(1/4)(6.79166667)+(0)(4.125)+(-1/48)(2.5)+(0)(1.5)+(1/1440)(1) = 18.42777778$$

Therefore,

$$\frac{b_6}{b_7} = \frac{11.1875}{18.42777778} = 0.60709979$$

Hence, the smallest root of the equation correct upto 4 decimal places is

$$x \approx 0.607$$

44. Using Ramanujan's method find the smallest root of

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \cdots = 0$$

correct upto 3 decimal places.

Given equation can be written as

$$1 - \left(x - \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} - \frac{x^4}{(4!)^2} + \dots\right) = 0$$

Comparing the R.H.S. of the equation with

$$1 - \left(a_1x + a_2x^2 + a_3x^3 + \dots\right)$$

We get the following,

$$a_1 = 1, a_2 = -1/4, a_3 = 1/36, a_4 = -1/576, a_5 = 1/14400, a_6 = -1/518400$$

We determine the constants b_i and the convergents $\frac{b_{i-1}}{b_i}$ as given below.

First take $b_1 = 1$

Then, for each sussessive b_k we shall use the following formula,

$$b_i = a_1 b_{i-1} + a_2 b_{i-2} + a_3 b_{i-3} + \cdots + +a_{i-1} b_1$$

Therefore,

$$b_2 = a_1 b_1 = a_1 = 1$$

Therefore,

$$\frac{b_1}{b_2} = \frac{1}{1} = 1$$

Now,

$$b_3 = a_1 b_2 + a_2 b_1$$

$$\therefore\ b_3 = (1)(1) + (-1/4)(1) = 0.75$$

Therefore,

$$\frac{b_2}{b_3} = \frac{1}{0.75} = 1.33333333$$

Now,

$$b_4 = a_1 b_3 + a_2 b_2 + a_3 b_1$$

$$\therefore\ b_4 = (1)(0.75) + (-1/4)(1) + (1/36)(1) = 0.52777778$$

Therefore,

$$\frac{b_3}{b_4} = \frac{0.75}{0.52777778} = 1.42105263$$

Now,

$$b_5 = a_1 b_4 + a_2 b_3 + a_3 b_2 + a_4 b_1$$

$$\therefore\ b_5 = (1)(0.52777778) + (-1/4)(0.75) + (1/36)(1) + (-1/576)(1) = 0.36631944$$

Therefore,

$$\frac{b_4}{b_5} = \frac{0.52777778}{0.36631944} = 1.44075829$$

Now,

$$b_6 = a_1 b_5 + a_2 b_4 + a_3 b_3 + a_4 b_2 + a_5 b_1$$

$$\therefore\ b_6 = (1)(0.36631944)+(-1/4)(0.52777778)+(1/36)(0.75)+(-1/576)(1)+(1/14400)(1) = 0.25354167$$

Therefore,

$$\frac{b_5}{b_6} = \frac{0.36631944}{0.25354167} = 1.44480964$$

Now,

$$b_7 = a_1 b_6 + a_2 b_5 + a_3 b_4 + a_4 b_3 + a_5 b_2 + a_6 b_1$$

$$\therefore\ b_7 = (1)(0.25354167)+(-1/4)(0.36631944)+(1/36)(0.52777778)+(-1/576)(0.75)+(1/14400)(1) +(-1/518400)(1) = 0.17538773$$

Therefore,

$$\frac{b_6}{b_7} = \frac{0.25354167}{0.17538773} = 1.44560663$$

Now,

$$b_8 = a_1 b_7 + a_2 b_6 + a_3 b_5 + a_4 b_4 + a_5 b_3 + a_6 b_2$$

$$\therefore\ b_8 = (1)(0.17538773)+(-1/4)(0.25354167)+(1/36)(0.36631944)+(-1/576)(0.52777778)+(1/14400)(0.75) +(-1/518400)(1) = 0.12131173$$

Therefore,

$$\frac{b_7}{b_8} = \frac{0.17538773}{0.12131173} = 1.44576072$$

Hence, the smallest root of the equation correct upto 3 decimal places is

$$x \approx 1.445$$

45. Absolute, Relative and Percentage errors.

Absolute, Relative and Percentage errors.

(1) ABSOLUTE ERROR : If X is the true value of a quantity and X_1 is its approximate value then difference between X and X_1 given by $\delta X = X - X_1$ is called the Ablosute error in X which is generally denoted by E_A.

$$E_A = X - X_1 = \delta X$$

(2) RELATIVE ERROR : If X is the ture value of a quantity and $E_A = \delta X$ is the absolute error then the Relative Error, generally denoted by E_R is defined by,

$$E_R = \frac{E_A}{X} = \frac{\delta X}{X}$$

(3) PERCENTAGE ERROR : If X is the ture value of a quantity and $E_A = \delta X$ is the absolute error then the percentage error, generally denoted by, E_P is given by,

$$E_P = \frac{\delta X}{X} \times 100$$

Thus, if E_R is the relative error then,

$$E_P = 100 E_R$$

46. Absolute accuracy and Relative Accuracy.

Absolute accuracy and Relative Accuracy

Let X be the true value of a quantity and X_1 be its approximate value. Then $|X\_X_1|$ is known as the magnitude of error. If ΔX is a number such that,

$$|X_1 - X| \leqslant \Delta X$$

then ΔX it is said to measure Absolute Accuracy and $\dfrac{\Delta X}{|X|}$ measures the Relative Accuracy.

47. Find the relative error of the number 4.2 if both of its digits are correct.

Since both the digits of 4.2 are correct, the absolute error can be assumed to be $E_A = 0.05$

Hence, $E_R = \dfrac{E_A}{X} = \dfrac{0.05}{4.2} = 0.0119$

48. True value of a quantity is q is $X = 5.45845$ and its approximate value is $X_1 = 5.45875$. Find its absolute and relative errors.

Here, true value $X = 5.45845$ and approximate value is $X_1 = 5.45875$

Therefore, absolute error,

$$E_A = X - X_1 = 5.45845 - 5.45875 = -0.0003$$

and relative error,

$$E_R = \frac{E_A}{X} = \frac{-0.0003}{5.45845} = -0.0000549$$

49. An approximate value of π is $X_1 = 3.1428571$ and its true value is $X = 3.1415926$. Find its absolute and relative errors.

Here true value of π is assumed to be $X = 3.1415926$ and its approximate value $X_1 = 3.1428571$.

Therefore, absolute error,

$$E_A = X - X_1 = 3.1415926 - 3.1428571 = -0.0012645$$

Hence, relative error,

$$E_R = \frac{E_A}{X} = \frac{-0.0012645}{3.1415926} = -0.000402$$

50. Three approximate values of the number $\frac{1}{3}$ are given by $0.30, 0.33$ and 0.34. Which of these vales is the best approximation?

Magnitudes of errors in approximations $0.30, 0.33$ and 0.34 of $\frac{1}{3}$ are given by,

$$\left|\frac{1}{3} - 0.30\right| = \frac{1}{30}$$

$$\left|\frac{1}{3} - 0.33\right| = \frac{1}{300}$$

$$\left|\frac{1}{3} - 0.34\right| = \frac{1}{150}$$

As the minimum magnitude in error $\frac{1}{300}$ corresponds to the approximation 0.33, it is the best approximation for $\frac{1}{3}$.

51. General Error Formula:

If $u = f(x_1, x_2, \ldots, x_n)$ and the error in each x_i is Δx_i then correspoinding error Δu in u is given by,

$$\delta u = \frac{\partial u}{\partial x_1}\Delta x_1 + \frac{\partial u}{\partial x_2}\Delta x_2 + \cdots + \frac{\partial u}{\partial x_n}\Delta x_n$$

52. **If $u = \dfrac{5xy^2}{z^3}$ then find the relative maximum error at (1,1,1) if the error** $\Delta x = \Delta y = \Delta z = 0.001$

For $u = \dfrac{5xy^2}{z^3}$ we have,

$$\frac{\partial u}{\partial x} = \frac{5y^2}{z^3}, \qquad \frac{\partial u}{\partial y} = \frac{10xy}{z^3}, \qquad \frac{\partial u}{\partial z} = -\frac{15xy^2}{z^4}$$

As,

$$\Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

we get,

$$\Delta u = \frac{5y^2}{z^3}\Delta x + \frac{10xy}{z^3}\Delta y - \frac{15xy^2}{z^4}\Delta z$$

Therefore, maximum possible value of Δu,

$$(\Delta u)_{max} = \left|\frac{5y^2}{z^3}\right| |\Delta x| + \left|\frac{10xy}{z^3}\right| |\Delta y| + \left|\frac{15xy^2}{z^4}\right| |\Delta z|$$

For $(x, y, z) = (1, 1, 1)$ and $\Delta x = \Delta y = \Delta z = 0.001$, we have,

$$(\Delta u)_{max} = (5)(0.001) + (10)(0.001) + (15)(0.001) = 0.03$$

and

$$u = 5$$

Therefore, relative maximum error is given by

$$(E_R)_{max} = \frac{(\Delta u)_{max}}{u} = \frac{0.003}{5} = 0.006$$