S.Y.B.Sc. : Semester - III

US03CMTH21

Numerical Methods

[Syllabus effective from June , 2019]

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US03CMTH21- UNIT : III

1. **Lagrange's Interpolation formula For data set**

x	4	5	7
y	10	-5	2

in which the values of x_i are not necessarily equally spaced, the Largrange's Interpolation formula to evaluate y at a given value of x is given by,

$$L_n(x) = \sum_{i=0}^{n} \frac{(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{i-1})(x-x_{i+1})\dots(x-x_n)}{(x_i-x_0)(x_i-x_1)(x_i-x_2)\dots(x_i-x_{i-1})(x_i-x_{i+1})\dots(x_i-x_n)} y_i$$

2. **Using Langrange's interpolation formula, find the form of the function $y(x)$ from the following table**

x	0	1	3	4
y	-12	0	12	24

Answer:
In the given set of data

x	0	1	3	4
y	-12	0	12	24

we note that $y(1) = 0$
Therefore $(x-1)$ is one of the factors of $y(x)$
Suppose, $y(x) = (x-1)L(x)$
then for the given data, excluding $x = 1$ we have

$$R(x) = \frac{y(x)}{x-1}$$

We obtain new table for $(x, R(x))$

x	0	3	4
$R(x) = \frac{y(x)}{x-1}$	$\frac{-12}{0-1} = 12$	$\frac{12}{3-1} = 6$	$\frac{24}{4-1} = 8$

Spacing between consecutive argumants are as follows
$3 - 0 = 3$
$4 - 3 = 1$

As the arguments are unevenly spaced, we shall use Lagrange's Interpolation formula

$$
\begin{aligned}
L(x) &= \frac{(x-3)(x-4)}{(0-3)(0-4)}(12) + \frac{(x-0)(x-4)}{(3-0)(3-4)}(6) + \frac{(x-0)(x-3)}{(4-0)(4-3)}(8) \\
&= (x^2-7x+12) - 2(x^2-4x) + 2(x^2-3x) \\
&= x^2-7x+12-2x^2+8x+2x^2-6x \\
&= x^2-5x+12
\end{aligned}
$$

$\therefore y(x) = (x-1)(x^2-5x+12)$

3. **Certain corresponding values of x and $\log_{10} x$ are**

$(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$, **and** $(307, 2.4871)$

Find $\log_{10}(301)$.

Answer:

For the given data

X	300	304	305	307
Y	2.4771	2.4829	2.4843	2.4871

We have to find $y(301)$

Spacing between consecutive arguments are as follows

$304-300=4$

$305-304=1$

$307-305=2$

As the arguments are unevenly spaced, we shall use Lagrange's Interpolation formula

$$
\begin{aligned}
L_n(x) =& \frac{(x-304)(x-305)(x-307)}{(300-304)(300-305)(300-307)}(2.4771) \\
&+ \frac{(x-300)(x-305)(x-307)}{(304-300)(304-305)(304-307)}(2.4829) \\
&+ \frac{(x-300)(x-304)(x-307)}{(305-300)(305-304)(305-307)}(2.4843) \\
&+ \frac{(x-300)(x-304)(x-305)}{(307-300)(307-304)(307-305)}(2.4871)
\end{aligned}
$$

$\therefore$

$$L_n(301) = \frac{(301-304)(301-305)(301-307)}{(300-304)(300-305)(300-307)}(2.4771) + \frac{(301-300)(301-305)(301-307)}{(304-300)(304-305)(304-307)}(2.4829) + \frac{(301-300)(301-304)(301-307)}{(305-300)(305-304)(305-307)}(2.4843) + \frac{(301-300)(301-304)(301-305)}{(307-300)(307-304)(307-305)}(2.4871)$$

Therefore,

$$L_n(301) = 2.478597143$$

4. **Using Langrage's interpolation formula express the following function as a sum of partial fractions**

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}$$

Answer:

We want to express $\frac{3\,x^2+x+1}{(x-1)\,(x-2)\,(x-3)}$ as a sum of partial fractions

Factors of the denominator are $(x-1)\,(x-2)\,(x-3)$
and the roots of the equation $(x-1)\,(x-2)\,(x-3)=0$ are 1 2 3
Using the numerator expression $3\,x^2+x+1$ we define

$$y(x) = 3\,x^2+x+1$$

Substituting roots for x in $y(x)$ we get the following table

x	1	2	3
y(x)	5	15	31

We shall use Lagrange's Interpolation formula to express $y(x)$,

$$y(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$\therefore 3\,x^2+x+1 = \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

Dividing both the sides with $(x-1)(x-2)(x-3)$ we get the partial fraction decomposition,

$$\frac{3\,x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2}\left(\frac{1}{x-1}\right) - 15\left(\frac{1}{x-2}\right) + \frac{31}{2}\left(\frac{1}{x-3}\right)$$

5. **Using Langrange's interpolation formula express the following function as a sum of partial fractions**

$$\frac{x^2+x-3}{x^3-2x^2-x+2}$$

Answer:

We want to express $\dfrac{x^2+x-3}{x^3-2\,x^2-x+2}$ as a sum of partial fractions

Factors of the denominator are $(x-1)\ (x+1)\ (x-2)$
and the roots of the equation $x^3-2\,x^2-x+2=0$ are -1 1 2
Using the numerator expression x^2+x-3 we define

$$y(x)=x^2+x-3$$

Substituting roots for x in $y(x)$ we get the following table

x	-1	1	2
y(x)	-3	-1	3

We shall use Lagrange's Interpolation formula to express y(x)

$$y(x)=\frac{(x-1)(x-2)}{(-1-1)(-1-2)}(-3)+\frac{(x+1)(x-2)}{(1+1)(1-2)}(-1)+\frac{(x+1)(x-1)}{(2+1)(2-1)}(3)$$

$$\therefore\quad x^2+x-3=-\frac{1}{2}(x-1)(x-2)+\frac{1}{2}(x+1)(x-2)+1(x+1)(x-1)$$

Dividing both the sides with $(x-1)\ (x+1)\ (x-2)$, we get partial fraction decomposition

$$\frac{x^2+x-3}{x^3-2\,x^2-x+2}=-\frac{1}{2}\left(\frac{1}{x+1}\right)+\frac{1}{2}\left(\frac{1}{x-1}\right)+1\left(\frac{1}{x-2}\right)$$

6. **Using Langrange's interpolation formula express the following function as a sum of partial fractions**

$$\frac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$$

Answer:

We want to express $\dfrac{x^2+6\,x+1}{(x-1)\ (x+1)\ (x-4)\ (x-6)}$ as a sum of partial fractions

Factors of the denominator are $(x-1)\ (x+1)\ (x-4)\ (x-6)$
and the roots of the equation $(x-1)\ (x+1)\ (x-4)\ (x-6)=0$ are -1 1 4 6
Using the numerator expression $x^2+6\,x+1$ we define

$$y(x)=x^2+6\,x+1$$

Substituting roots for x in $y(x)$ we get the following table

x	-1	1	4	6
y(x)	-4	8	41	73

We shall use Lagrange's Interpolation formula to express y(x)

$$
\begin{aligned}
y(x) =& \frac{(x-1)(x-4)(x-6)}{(-1-1)(-1-4)(-1-6)}(-4) \\
&+ \frac{(x+1)(x-4)(x-6)}{(1+1)(1-4)(1-6)}(8) \\
&+ \frac{(x+1)(x-1)(x-6)}{(4+1)(4-1)(4-6)}(41) \\
&+ \frac{(x+1)(x-1)(x-4)}{(6+1)(6-1)(6-4)}(73) \\
x^2+6x+1 =& \frac{2}{35}(x-1)(x-4)(x-6) \\
&+ \frac{4}{15}(x+1)(x-4)(x-6) \\
&- \frac{41}{30}(x+1)(x-1)(x-6) \\
&+ \frac{73}{70}(x+1)(x-1)(x-4)
\end{aligned}
$$

Dividing both the sides with $(x-1)\ (x+1)\ (x-4)\ (x-6)$, we get partial fraction decomposition

$$\frac{x^2+6x+1}{(x-1)\ (x+1)\ (x-4)\ (x-6)} = \frac{2}{35}\left(\frac{1}{x+1}\right) + \frac{4}{15}\left(\frac{1}{x-1}\right) - \frac{41}{30}\left(\frac{1}{x-4}\right) + \frac{73}{70}\left(\frac{1}{x-6}\right)$$

7. Divided Differences

Divided Differences:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given $n+1$ points. Then the divided differences of orders $1, 2, \ldots, n$ are difined by the relations.

$$
\begin{aligned}
[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\
[x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \\
[x_0, x_1, x_2, x_3] &= \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \\
&\vdots \\
[x_0, x_1, x_2 \ldots, x_n] &= \frac{[x_1, x_2, \ldots, x_n] - [x_0, x_1, \ldots, x_{n-1}]}{x_n - x_0}
\end{aligned}
$$

8. Show that the divided differences are symetrical in their arguments

Answer:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given $n+1$ points. Then the first order divided difference $[x_0, x_1]$ is

$$\begin{aligned}[x_0, x_1] &= \frac{y_1 - y_0}{x_1 - x_0} \\ &= \frac{y_1}{x_1 - x_0} - \frac{y_0}{x_1 - x_0} \\ &= \frac{y_0}{x_0 - x_1} + \frac{y_1}{x_1 - x_0} \\ &= [x_1, x_0]\end{aligned}$$

Again

$$\begin{aligned}[x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \\ &= \frac{\left(\frac{y_2 - y_1}{x_2 - x_1}\right) - \left(\frac{y_1 - y_0}{x_1 - x_0}\right)}{x_2 - x_0} \\ &= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} - y_1\left(\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0}\right) + \frac{y_0}{x_1 - x_0}\right] \\ &= \frac{1}{x_2 - x_0}\left[\frac{y_2}{x_2 - x_1} - y_1\left(\frac{x_2 - x_0}{(x_2 - x_1)(x_1 - x_0)}\right) + \frac{y_0}{x_1 - x_0}\right] \\ &= \frac{y_2}{(x_2 - x_0)(x_2 - x_1)} + \frac{y_1}{(x_1 - x_2)(x_1 - x_0)} + \frac{y_0}{(x_1 - x_0)(x_2 - x_0)} \\ &= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}\end{aligned}$$

Similarly it can be shown that

$$\begin{aligned}[x_0, x_1, x_2, \ldots, x_n] = \frac{y_0}{(x_0 - x_1)\ldots(x_0 - x_n)} + \frac{y_1}{(x_1 - x_0)\ldots(x_1 - x_n)} + \\ \cdots + \frac{y_n}{(x_n - x_0)\ldots(x_n - x_{n-1})}\end{aligned}$$

Hence the divided differences are symmetrical in their arguments.

9. In usual notations prove that $[x_0, x_1, x_2, x_3, \ldots, x_n] = \frac{1}{h^n.n!}\Delta^n y_0$

Answer:

[Using Principle of Mathematical Induction]

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ be given $n+1$ points which are equally spaced at a distance h.
Then $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots = x_n - x_{n-1} = h$

Now, we have the first order difference $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{\Delta y_0}{h}$

and in general we get, $[x_{n-1}, x_n] = \dfrac{y_n - y_{n-1}}{x_n - x_{n-1}} = \dfrac{\Delta y_{n-1}}{h}$

Also,

$$\begin{aligned}[x_0, x_1, x_2] &= \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0} \\ &= \frac{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}}{2h} \\ &= \frac{\Delta y_1 - \Delta y_0}{2h^2} \\ &= \frac{\Delta^2 y_0}{2! h^2}\end{aligned}$$

Now, let us assume that the result is true for a positive integer $n = k$. i.e.

$$[x_0, x_1, x_2, x_3, \dots, x_{k-1}, x_k] = \frac{1}{h^k . n!} \Delta^k y_0$$

Now, for $n = k+1$, using the definition of Newton's Divided difference of k^{th} order,

$$\begin{aligned}[x_0, x_1, x_2, x_3, \dots, x_k, x_{k+1}] &= \frac{[x_1, x_1, x_2, x_3, \dots, x_k, x_{k+1}] - [x_0, x_1, x_2, x_3, \dots, x_k]}{x_{k+1} - x_0} \\ &= \frac{\dfrac{\Delta^k y_1}{k! h^k} - \dfrac{\Delta^k y_0}{k! h^k}}{(k+1)h} \\ &= \frac{\Delta^k y_1 - \Delta^k y_0}{(k+1) k! h^{k+1}} \\ \therefore\ [x_0, x_1, x_2, x_3, \dots, x_k, x_{k+1}] &= \frac{\Delta^{k+1} y_0}{(k+1)! h^{k+1}}\end{aligned}$$

Thus, the result is true for $k+1$ whenever it is true for k.

Therefore, the principle of mathematical induction, it is true for all postive integers.

$$[x_0, x_1, x_2, x_3, \dots, x_n] = \frac{1}{h^n . n!} \Delta^n y_0, \qquad \forall\, n \in N$$

10. Derive Newton's divided difference formula

Answer:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be given $n+1$ points corrsponding to a function $y(x)$. Now, by the definition of first order divided difference we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

Therefore,

$$(x - x_0)[x, x_0] = y - y_0$$

Hence,

$$y = y_0 + (x - x_0)[x, x_0] - - - (1)$$

Also,

$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

Therefore,

$$[x, x_0, x_1](x - x_1) = [x, x_0] - [x_0, x_1]$$

Hence,

$$[x, x_0] = [x_0, x_1] + [x, x_0, x_1](x - x_1)$$

Substituting in (1) we get,

$$y = y_0 + (x - x_0)([x_0, x_1] + [x, x_0, x_1](x - x_1))$$
$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] - - - (2)$$

Again,

$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0, x_1, x_2]}{x - x_2}$$

Therefore,

$$[x, x_0, x_1, x_2](x - x_2) = [x, x_0, x_1] - [x_0, x_1, x_2]$$

Hence,

$$[x, x_0, x_1] = [x_0, x_1, x_2] + [x, x_0, x_1, x_2](x - x_2)$$

Substituting in (2) we get,

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)([x_0, x_1, x_2] + [x, x_0, x_1, x_2](x - x_2))$$
$$y = y_0 + (x - x_0)([x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2])$$

Proceeding in this way we get,

$$\begin{aligned} y = y_0 &+ (x - x_0)[x_0, x_1] \\ &+ (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] \\ &+ \cdots + (x - x_0)(x - x_1)\ldots(x - x_n)[x, x_0, x_1, x_2, \ldots, x_n] \end{aligned}$$

This is known as Newton's divided difference formula.

11. **Using Newton's divided difference formula find $f(x)$ as a polynomial in x from the following table. Also find $f(4)$**

x	-1	0	3	6	7
y	3	-6	39	822	1611

Answer:

For the given data

X	-1	0	3	6	7
Y	3	-6	39	822	1611

We have to find $y(4)$

Following is the Divided difference table of the data

-1	3				
		-9			
0	-6		6		
		15		5	
3	39		41		1
		261		13	
6	822		132		
		789			
7	1611				

We shall use Newton's Divided Difference Interpolation formula

$$y_x = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] +$$

$$... + (x-x_0)(x-x_1)(x-x_2)...(x-x_n)[x,x_0,x_1,...x_n]$$

Substituting for the divided differences and $x_0, x_1, x_2 \ldots$ we get

$$y = 3 + (x+1)(-9) + x\ (x+1)\,(6) + x\ (x+1)\ (x-3)\,(5) + x\ (x+1)\ (x-3)\ (x-6)\,(1)$$

Therefore

$$y = 3 + (-9\,x - 9) + (6\,x^2 + 6\,x) + (5\,x^3 - 10\,x^2 - 15\,x) + (x^4 - 8\,x^3 + 9\,x^2 + 18\,x)$$

Hence

$$y = x^4 - 3\,x^3 + 5\,x^2 - 6$$

Taking $x = 4$ we get,$y(4) = 138$

12. **Given the set of tabulated points (x,y) which are $(1,-3)$, $(3,9)$, $(4,30)$ and $(6,132)$ obtain the value of y when $x = 2$ using Newton's divided difference formula**

Answer:

For the given data

X	1	3	4	6
Y	-3	9	30	132

We have to find $y(2)$

Following is the Divided difference table of the data

1	-3			
		6		
3	9		5	
		21		1
4	30		10	
		51		
6	132			

We shall use Newton's Divided Difference Interpolation formula

$$y_x = y_0 + (x-x_0)[x_0,x_1] + (x-x_0)(x-x_1)[x_0,x_1,x_2] + (x-x_0)(x-x_1)(x-x_2)[x_0,x_1,x_2,x_3] +$$

$$... + (x-x_0)(x-x_1)(x-x_2)...(x-x_n)[x,x_0,x_1,...x_n]$$

Substituting $x = 2,\ y_0$, the divided differences and values of $x_0, x_1, x_2 \ldots$ from the table we get,

$$y(2) = -3 + (2-1)(6) + (2-1)(2-3)(5) + (2-1)(2-3)(2-4)(1)$$

Therefore $y(2) = 0$

13. If $y_1 = 4,\ y_3 = 12,\ y_4 = 19$ and $y_x = 7$ find x. Write the formula you use and also give it's name

Answer:

For the given data

y	4	12	19
x	1	3	4

We have to find $x(7)$

Spacing between consecutive arguments are as follows

$12 - 4 = 8$
$19 - 12 = 7$

As the arguments are unevenly spaced, we shall use Lagrange's Interpolation formula

$$x(y) = \frac{(y-12)(y-19)}{(4-12)(4-19)}(1) + \frac{(y-4)(y-19)}{(12-4)(12-19)}(3) + \frac{(y-4)(y-12)}{(19-4)(19-12)}(4)$$

Therefore,

$$x(7) = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

Therefore,

$$x(7) = 1.857142857$$

14. Inverse Interpolation.

Inverse Interpolation:

Let $(x_0,y_0),(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ be $(n+1)$ points corresponding to an unknown function y of x.

A process of finding the value of x for a given value of y is called inverse interpolation.

15. Discuss the method of succesive apprximation for inverse interpolation.

method of succesive apprximation for inverse interpolation:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be $(n+1)$ points corresponding to an unknown function y of x.
We assume that the argument $x_0, x_1, x_2 \ldots, x_n$ are equally spaced at a distance of h.
Suppose for a given value of y corresponding x is given by.

$$x = x_0 + uh$$

for some u.
To find a sufficiently accurate value of x we shall find sufficiently accurate value of u. We shall start with Newton's forward difference interpolation formula.

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \cdots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0$$

As we wish to find u, let us express the formula as

$$u = \frac{1}{\Delta y_0}\left[y - y_0 - \frac{u(u-1)}{2!}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 - \cdots - \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0\right]$$

- - - - (1)

Neglecting the second and higher order differences we get first approximation u_1 of u as,

$$u_1 = \frac{1}{\Delta y_0}(y - y_0)$$

Next we shall retain terms upto second order differences in (1) and obtain second approximation u_2 by substituting u_1 for u,

$$u_2 = \frac{1}{\Delta y_0}\left[y - y_0 - \frac{u_1(u_1-1)}{2!}\Delta^2 y_0\right]$$

Now we shall retain terms upto third order differences in (1) and obtain third approximation u_3 by substituting u_2 for u,

$$u_3 = \frac{1}{\Delta y_0}\left[y - y_0 - \frac{u_2(u_2-1)}{2!}\Delta^2 y_0 - \frac{u_2(u_2-1)(u_2-2)}{3!}\Delta^3 y_0\right]$$

The process can be continued for obtaining approximations $u_1, u_2, u_3 \ldots$ successively and we can stop when two successive approximations of u agree with each other to the required accuracy.

16. Tabulate $y = x^3$ for $x = 2, 3, 4, 5$ and calculate $\sqrt[3]{10}$ correct upto three decimal places

Answer:

To find cube root of 10 by tabulating values of $y = x^3$ for $x = 2, 3, 4, 5$ we shall interpolate inversely for x corresponding to $y = 10$

Now using $y = x^3$ for $x = 2, 3, 4, 5$ gives following table

X	2	3	4	5
Y	8	27	64	125

Following is the difference table of the data

2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

Here, we have h = 3 - 2 = 1

Since, given value of y is near 8, which corresponds the first argument, for x corresponding to y we shall take, $x = x_0 + uh$

For finding successive approximations, we shall use a formula based on Newton's forward difference interpolation formula given below

$$u = \frac{1}{\Delta y_0}(y - y_0 - \frac{u(u-1)}{2!}\Delta^2 y_0 - \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 - \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 - \frac{u(u-1)(u-2)(u-3)(u-4)}{5!}\Delta^5 y_0 - ...)$$

and obtain the successive approximations as follows

Using $u_1 = \frac{1}{\Delta y_0}[y - y_0]$, we get
$u_1 = \frac{1}{19}[10 - 8] = 0.1052631579$

Using $u_2 = \frac{1}{\Delta y_0}[y - y_0 - \frac{\text{u1 (u1−1)}}{2}\Delta^2 y_0]$, we get
$u_2 = \frac{1}{19}[10 - 8 - (-0.04709141274)18] = 0.1498760752$

Using $u_3 = \frac{1}{\Delta y_0}[y - y_0 - \frac{\text{u2 (u2−1)}}{2}\Delta^2 y_0 - \frac{\text{u2 (u2−1) (u2−2)}}{6}\Delta^3 y_0]$, we get
$u_3 = \frac{1}{19}[10 - 8 - (-0.06370661865)18 - (0.03928837978)6] = 0.1532099398$

Using $u_4 = \frac{1}{\Delta y_0}[y - y_0 - \frac{\text{u3 (u3−1)}}{2}\Delta^2 y_0 - \frac{\text{u3 (u3−1) (u3−2)}}{6}\Delta^3 y_0]$, we get
$u_4 = \frac{1}{19}[10 - 8 - (-0.06486832709)18 - (0.03993272723)6] = 0.1541070276$

Using $u_5 = \frac{1}{\Delta y_0}[y - y_0 - \frac{\text{u4 (u4−1)}}{2}\Delta^2 y_0 - \frac{\text{u4 (u4−1) (u4−2)}}{6}\Delta^3 y_0]$, we get
$u_5 = \frac{1}{19}[10 - 8 - (-0.06517902582)18 - (0.0401045019)6] = 0.1543471291$

Here 0.1543471291 and 0.1541070276 both agree upto 3 digits accuracy after the decimal points
Thus, using $x = x_0 + uh$, we get

$$x = 2 + (0.154)(1) = 2.154$$

which is correct upto 3 decimal places.

17. Obtain 1^{st} and 2^{nd} order numerical differentiation formula from Newton's forward difference formula

Answer:
Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be $(n+1)$ points corresponding to an unknown function y of x, where $x_0, x_1, x_2 \ldots, x_n$ are equally spaced at a distance of h.
For some x near x_0 suppose $x = x_0 + uh$

To find y corresponding to x we have the Newton's forward difference formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \cdots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0 \quad \text{--- (1)}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

Therefore from (1) we get,

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$= \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2}\Delta^2 y_0 + \frac{3u^2-6u+2}{6}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{24}\Delta^4 y_0 + \ldots\right] \quad \text{--- (2)}$$

This formula can be used for finding approximate derivative at a non-tabulated value of x

In case x is a tabulated value then setting $x = x_0$ we have $u = 0$.

Substituting 0 for u in (2) we get

$$\left[\frac{dy}{dx}\right]_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right] \quad \text{--- (3)}$$

which can be used to find derivative of a tabulated value x.
Also differentiating (2) with respect to x we get

$$\frac{d^2y}{dx^2} = \frac{1}{h}\frac{d^2y}{du^2}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + \frac{6u-6}{6}\Delta^3 y_0 + \frac{12u^2-36u+22}{24}\Delta^4 y_0 + \ldots\right]$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \ldots\right] --- (4)$$

This formula can be used for finding approximate second derivative at a non-tabulated value of x

In case x is a tabulated value then setting $x = x_0$ we have $u = 0$.

Substituting 0 for u in (4) we get,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \frac{137}{180}\Delta^6 + \ldots\right] --- (5)$$

which can be used to find second derivative of a tabulated value x.

18. Obtain 1^{st} and 2^{nd} order numerical differentiation formula from Newton's backward difference formula

Answer:

Let $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ be $(n+1)$ points corresponding to an unknown function y of x, where $x_0, x_1, x_2 \ldots, x_n$ are equally spaced at a distance of h.
For some x near x_n suppose $x = x_n + uh$

To find y corresponding to x we have the Newton's backward difference formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n + \cdots + \frac{u(u+1)\ldots(u+n-1)}{n!}\nabla^n y_n \quad \text{--- (1)}$$

Now,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{h}\frac{dy}{du}$$

Therefore from (1) we get,

$$\frac{dy}{dx} = \frac{1}{h}\frac{dy}{du}$$

$$= \frac{1}{h}\left[\nabla y_n + \frac{2u+1}{2}\nabla^2 y_n + \frac{3u^2+6u+2}{6}\nabla^3 y_n + \frac{4u^3+18u^2+22u+6}{24}\nabla^4 y_n + \ldots\right] \quad \text{--- (2)}$$

This formula can be used for finding approximate derivative at a non-tabulated value of x

In case x is a tabulated value then setting $x = x_n$ we have $u = 0$.

Substituting 0 for u in (2) we get

$$\left[\frac{dy}{dx}\right]_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right] \quad \text{--- (3)}$$

which can be used to find derivative of a tabulated value x.
Also differentiating (2) with respect to x we get

$$\frac{d^2y}{dx^2} = \frac{1}{h}\frac{d^2y}{du^2}$$
$$= \frac{1}{h^2}\left[\nabla^2 y_n + \frac{6u+6}{6}\nabla^3 y_n + \frac{12u^2+36u+22}{24}\nabla^4 y_n + \ldots\right]$$

Therefore,

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{6u^2+18u+11}{12}\nabla^4 y_n + \ldots\right] --- (4)$$

This formula can be used for finding approximate second derivative at a non-tabulated value of x

In case x is a tabulated value then setting $x = x_n$ we have $u = 0$.

Substituting 0 for u in (4) we get,

$$\left[\frac{d^2y}{dx^2}\right]_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \frac{137}{180}\nabla^6 y_n + \dots\right] \quad --- (5)$$

which can be used to find second derivative of a tabulated value x.

19. The following table of values of x and y is given :

x	0	1	2	3	4	5	6
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 6$

Answer:

For the given data

X	0	1	2	3	4	5	6
Y	6.9897	7.4036	7.7815	8.1291	8.451	8.7506	9.0309

We have to find $\frac{dy}{dx}$ and $\frac{d2y}{dx2}$ at $x = 6$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$
0	6.9897						
		0.4139					
1	7.4036		-0.036				
		0.3779		0.0057			
2	7.7815		-0.0303		-0.0011		
		0.3476		0.0046		-0.0001	
3	8.1291		-0.0257		-0.0012		0.0009
		0.3219		0.0034		0.0008	
4	8.451		-0.0223		-0.0004		
		0.2996		0.003			
5	8.7506		-0.0193				
		0.2803					
6	9.0309						

Here, we have h = 1 - 0 = 1

As, $x_0 = 6$ is one of the arguments in the second half of the data, we shall use the following numerical diffrentiation formula derived from

Newton's Backward Difference Interpolation formula

$\frac{dy}{dx} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + ...]$Therefore

$$\frac{dy}{dx} = \frac{1}{1}\left[1(0.2803) + \frac{1}{2}(-0.0193) + \frac{1}{3}(0.003) + \frac{1}{4}(-0.0004) + \frac{1}{5}(0.0008) + \frac{1}{6}(0.0009)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=6} = 0.27186$$

and
$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + ...]$
Therefore

$$\frac{d^2y}{dx^2} = \frac{1}{1}\left[+1(-0.0193) + 1(0.003) + \frac{11}{12}(-0.0004) + \frac{5}{6}(0.0008) + \frac{137}{180}(0.0009)\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=6} = -0.015315$$

Thus,

$$\left[\frac{dy}{dx}\right]_{x=6} = 0.27186 \quad \text{and} \quad \left[\frac{d^2y}{dx^2}\right]_{x=6} = -0.015315$$

20. **The following table of values of x and y is given :**

x	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	**2.2**
y	**2.7183**	**3.3201**	**4.0552**	**4.9530**	**6.0496**	**7.3891**	**9.0250**

Find first and second derivatives of y w.r.t. x when $x = 1.2$

Answer:
For the given data

X	1	1.2	1.4	1.6	1.8	2	2.2
Y	2.7183	3.3201	4.0552	4.953	6.0496	7.3891	9.025

We have to find $\frac{dy}{dx}$ and $\frac{d2y}{dx2}$ at $x = 1.2$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$	$\Delta^6 Y$
1	2.7183						
		0.6018					
1.2	3.3201		0.1333				
		0.7351		0.0294			
1.4	4.0552		0.1627		0.0067		
		0.8978		0.0361		0.0013	
1.6	4.953		0.1988		0.008		0.0001
		1.0966		0.0441		0.0014	
1.8	6.0496		0.2429		0.0094		
		1.3395		0.0535			
2	7.3891		0.2964				
		1.6359					
2.2	9.025						

Here, we have h = 1.2 - 1 = 0.2

As, $x_0 = 1.2$ is one of the arguments in the first half of the data, we shall use the following numerical diffrentiation formula

$$\frac{dy}{dx} = \frac{1}{h}[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + ...]$$

Therefore

$$\frac{dy}{dx} = \frac{1}{0.2}\left[1(0.7351) - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.008) + \frac{1}{5}(0.0014)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=1.2} = 3.320316667$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + ...]$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{1}{0.04}\left[+1(0.1627) - 1(0.0361) + \frac{11}{12}(0.008) - \frac{5}{6}(0.0014)\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=1.2} = 3.319166667$$

Thus,

$$\left[\frac{dy}{dx}\right]_{x=1.2} = 3.320316667 \quad \text{and} \quad \left[\frac{d^2y}{dx^2}\right]_{x=1.2} = 3.319166667$$

21. **Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 5$ for the following data :**

x	0	1	2	3	4	5
y	**6.98**	**7.40**	**7.78**	**8.12**	**8.45**	**8.75**

Answer:

For the given data

X	0	1	2	3	4	5
Y	6.98	7.4	7.78	8.12	8.45	8.75

We have to find $\frac{dy}{dx}$ and $\frac{d2y}{dx2}$ at $x = 5$

Following is the diffrence table of the data

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$	$\Delta^5 Y$
0	6.98					
		0.42				
1	7.4		-0.04			
		0.38		0.0		
2	7.78		-0.04		0.03	
		0.34		0.03		-0.08
3	8.12		-0.01		-0.05	
		0.33		-0.02		
4	8.45		-0.03			
		0.3				
5	8.75					

Here, we have h = 1 - 0 = 1

As, $x_0 = 5$ is one of the arguments in the second half of the data, we shall use the following numerical diffrentiation formula derived from

Newton's Backward Difference Interpolation formula

$$\frac{dy}{dx} = \frac{1}{h}[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n + ...]$$

Therefore

$$\frac{dy}{dx} = \frac{1}{1}\left[1(0.3) + \frac{1}{2}(-0.03) + \frac{1}{3}(-0.02) + \frac{1}{4}(-0.05) + \frac{1}{5}(-0.08)\right]$$

$$\left[\frac{dy}{dx}\right]_{x=5} = 0.2498333333$$

and

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + ...\ ]$$

Therefore

$$\frac{d^2y}{dx^2} = \frac{1}{1}\left[+1(-0.03) + 1(-0.02) + \frac{11}{12}(-0.05) + \frac{5}{6}(-0.08)\right]$$

$$\left[\frac{d^2y}{dx^2}\right]_{x=5} = -0.1625$$

Thus,

$$\left[\frac{dy}{dx}\right]_{x=5} = 0.2498333333 \quad \text{and} \quad \left[\frac{d^2y}{dx^2}\right]_{x=5} = -0.1625$$

22. From the following table, find x correct upto two decimal places ,for which y is maximum and find the value of y

x	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**
$y = f(x)$	**0.9320**	**0.9636**	**0.9855**	**0.9975**	**0.9996**

Answer:

For the given data

X	1.2	1.3	1.4	1.5	1.6
Y	0.932	0.9636	0.9855	0.9975	0.9996

Following is the diffrence table.

X	Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
1.2	0.932				
		0.0316			
1.3	0.9636		-0.0097		
		0.0219		-0.0002	
1.4	0.9855		-0.0099		0.0002
		0.012		0	
1.5	0.9975		-0.0099		
		0.0021			
1.6	0.9996				

We have to find the value of x such that y is maximum.

We know that for an extreme value of y we have $\frac{dy}{dx} = 0$