T.Y.B.Sc. : Semester - V

US05CMTH22(T)

Theory Of Real Functions

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US05CMTH22(T)- UNIT : II

1. Increasing Function at a point

Increasing Function at a point

Let f be a function defined in some neighbourhood of a number c. If there is some $\delta > 0$ such that

$$f(x) \leqslant f(c), \ \forall \, x \in (c - \delta, c)$$

and

$$f(c) \leqslant f(x), \ \forall \, x \in (c, c + \delta)$$

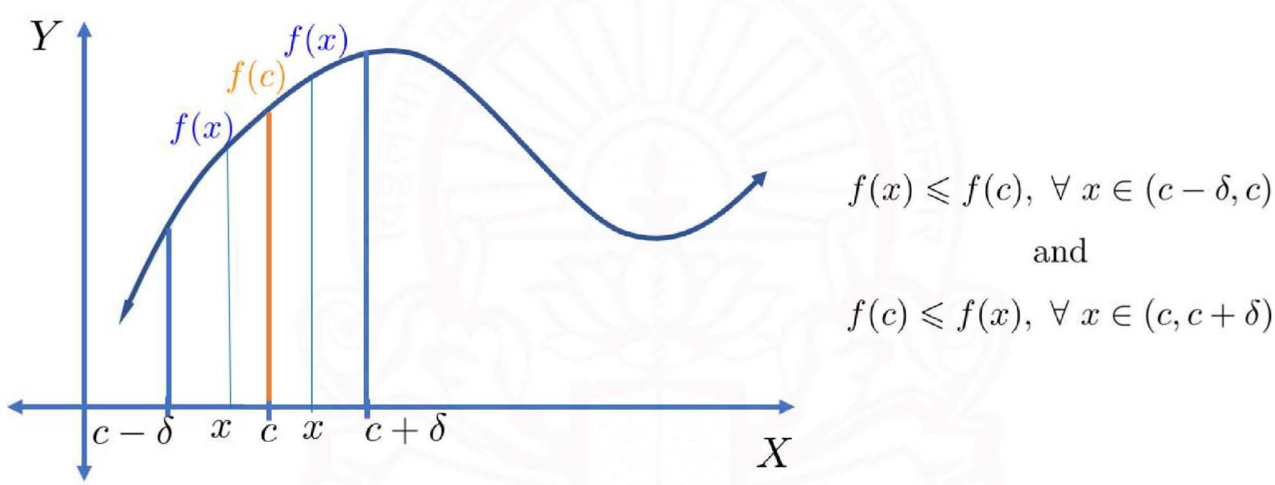


Figure 1: Increasing Function at a point c

then f is said to be an Increasing function at c.

2. Strictly Increasing Function at a point

Strictly Increasing Function at a point

Let f be a function defined in some neighbourhood of a number c. If there is some $\delta > 0$ such that

$$f(x) < f(c), \ \forall \, x \in \ (c - \delta, c)$$

and

$$f(c) < f(x), \ \forall \, x \in \ (c, c + \delta)$$

then f is said to be a Strictly Increasing function at c.

3. Decreasing Function at a point

Decreasing Function at a point

Let f be a function defined in some neighbourhood of a number c. If there is some $\delta > 0$ such that

$$f(x) \geqslant f(c), \ \forall \ x \in (c-\delta, c)$$

and

$$f(c) \geqslant f(x), \ \forall \ x \in (c, c+\delta)$$

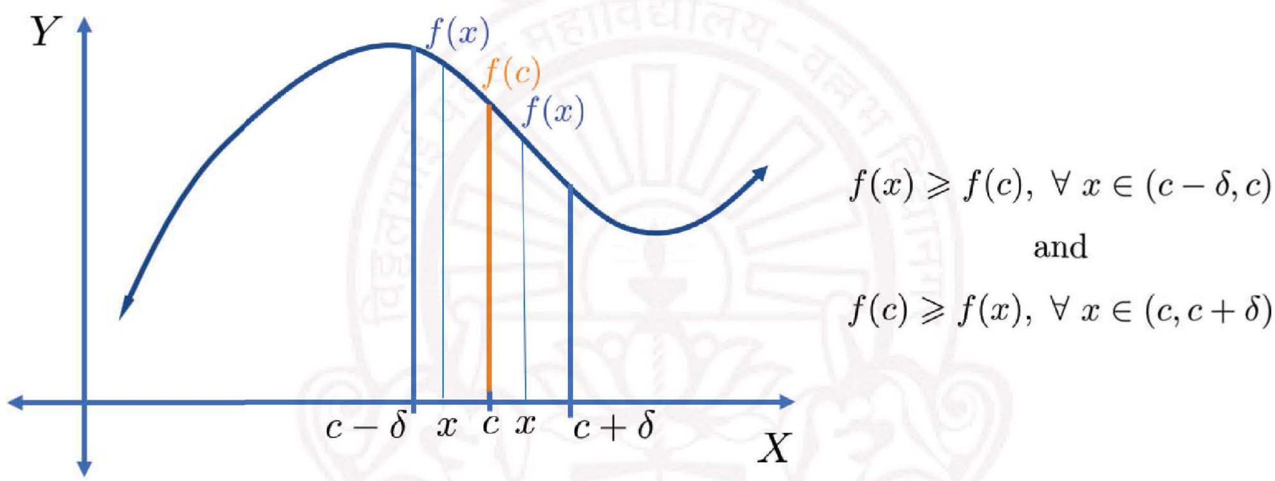


Figure 2: Decreasing Function at a point c

then f is said to be a Decreasing function at c.

4. Strictly Decreasing Function at a point

Strictly Decreasing Function at a point

Let f be a function defined in some neighbourhood of a number c. If there is some $\delta > 0$ such that

$$f(x) > f(c), \ \forall \ x \in \ (c-\delta, c)$$

and

$$f(c) > f(x), \ \forall \ x \in \ (c, c+\delta)$$

then f is said to be a Strictly Decreasing function at c.

5. Increasing Function in an interval.

Increasing Function in an interval

Let f be a function defined on an interval $[a, b]$.
If

$$f(x_1) \leqslant f(x_2),\ \forall\ x_1 \leqslant x_2, \quad \text{where} \ \ x_1, x_2 \in [a, b]$$

then f is said to be an Increasing function on $[a, b]$.

6. Strictly Increasing Function in an interval.

Strictly Increasing Function in an interval

Let f be a function defined on an interval $[a, b]$.
If

$$f(x_1) < f(x_2),\ \forall\ x_1 < x_2, \quad \text{where} \ \ x_1, x_2 \in [a, b]$$

then f is said to be a Strictly Increasing function on $[a, b]$.

7. Decreasing Function in an interval.

Decreasing Function in an interval

Let f be a function defined on an interval $[a, b]$.
If

$$f(x_1) \geqslant f(x_2),\ \forall\ \ x_1 \leqslant x_2 \quad \text{where} \ \ x_1, x_2 \in [a, b]$$

then f is said to be a Decreasing function on $[a, b]$.

8. Strictly Decreasing Function in an interval.

Strictly Decreasing Function in an interval

Let f be a function defined on an interval $[a, b]$.
If

$$f(x_1) > f(x_2),\ \forall\ x_1 < x_2, \quad \text{where} \ \ x_1, x_2 \in [a, b]$$

then f is said to be an Strictly Decreasing function on $[a, b]$.

9. If $f'(c) > 0$, then prove that f is an increasing function at point $x = c$.

Proof:

Let f be a function defined on $[a, b]$ such that it is derivable at a point $c \in (a, b)$.

We get the derivative by

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

For any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \text{ whenever } 0 < |x - c| < \delta$$

This implies that,

$$f'(c) - \epsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \epsilon \text{ whenever } x \in (c - \delta, c + \delta),\ x \neq c \quad \text{- - - (1)}$$

Now, suppose $f'(c) > 0$.

Then we can select some sufficiently small $\epsilon > 0$ such that $0 < f'(c) - \epsilon$.

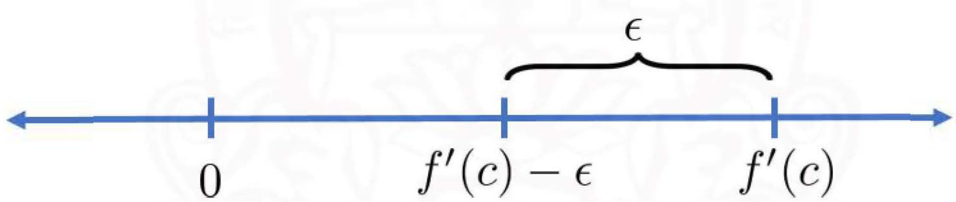


Figure 3: Selecting $\epsilon > 0$ such that $0 < f'(c) - \epsilon$

For this choice of $\epsilon > 0$ there must be some $\delta > 0$ satisfying (1).

As $0 < f'(c) - \epsilon$, from (1) it follows that,

$$0 < \frac{f(x) - f(c)}{x - c} \text{ whenever } x \in (c - \delta, c + \delta),\ x \neq c \quad \text{- - - (2)}$$

If $x \in (c - \delta, c)$ then $x - c < 0$. So to have $0 < \dfrac{f(x) - f(c)}{x - c}$ we must have

$$f(x) - f(c) < 0 \quad \text{whenever} \quad x \in (c - \delta, c)$$

This implies that,

$$f(x) < f(c) \quad \text{whenever} \quad x \in (c - \delta, c) \quad \text{- - - (3)}$$

Also, if $x \in (c, c + \delta)$ then $x - c > 0$. So to have $0 < \dfrac{f(x) - f(c)}{x - c}$ we must have

$$f(x) - f(c) > 0 \quad \text{whenever} \quad x \in (c, c + \delta)$$

This implies that,

$$f(c) < f(x) \quad \text{whenever} \quad x \in (c, c+\delta) \qquad \text{- - - } (4)$$

From (3) and (4) it follows that, f is increasing at c

10. If $f'(c) < 0$, then prove that f is a decreasing function at point $x = c$.

Proof:

Let f be a function defined on $[a, b]$ such that it is derivable at a point $c \in (a, b)$.

We get the derivative by

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

For any given $\epsilon > 0$ there exists some $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon \;\; \text{whenever} \;\; 0 < |x - c| < \delta$$

This implies that,

$$f'(c) - \epsilon < \frac{f(x) - f(c)}{x - c} < f'(c) + \epsilon \;\; \text{whenever} \;\; x \in (c - \delta, c + \delta), \; x \neq c \qquad \text{- - - } (1)$$

Now, suppose $f'(c) < 0$.

Then we can select some sufficiently small $\epsilon > 0$ such that $f'(c) + \epsilon < 0$.

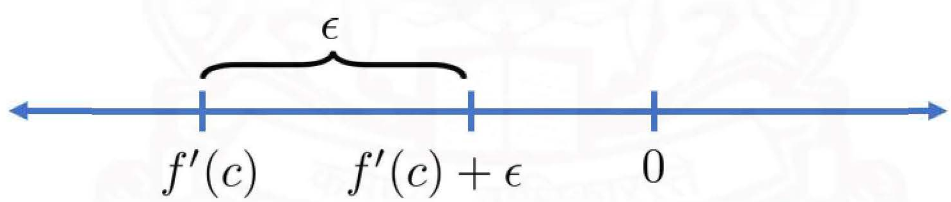


Figure 4: Selecting $\epsilon > 0$ such that $f'(c) + \epsilon < 0$

For this choice of $\epsilon > 0$ there must be some $\delta > 0$ satisfying (1).

As $f'(c) + \epsilon < 0$, from (1) it follows that,

$$\frac{f(x) - f(c)}{x - c} < 0 \;\; \text{whenever} \;\; x \in (c - \delta, c + \delta), \; x \neq c \qquad \text{- - - } (2)$$

If $x \in (c - \delta, c)$ then $x - c < 0$. So to have $\dfrac{f(x) - f(c)}{x - c} < 0$ we must have

$$f(x) - f(c) > 0 \quad \text{whenever} \quad x \in (c - \delta, c)$$

This implies that,

$$f(x) > f(c) \quad \text{whenever} \quad x \in (c - \delta, c) \qquad \text{- - - } (3)$$

Also, if $x \in (c, c+\delta)$ then $x - c > 0$. So to have $\dfrac{f(x)-f(c)}{x-c} < 0$ we must have

$$f(x) - f(c) < 0 \quad \text{whenever} \quad x \in (c, c+\delta)$$

This implies that,

$$f(x) < f(c) \quad \text{whenever} \quad x \in (c, c+\delta) \quad \text{- - - (4)}$$

From (3) and (4) it follows that, f is decreasing at c

11. **Show that** $\log(1+x)$ **lies between** $x - \dfrac{x^2}{2}$ **and** $x - \dfrac{x^2}{2(1+x)}$, $\forall x > 0$

Solution:

We have to prove that $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}$, $\forall x > 0$

First define, $f(x) = \log(1+x) - \left(x - \dfrac{x^2}{2}\right)$

Here,

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - (1-x) \\ &= \frac{1-(1-x^2)}{1+x} \\ &= \frac{x^2}{x+1} \\ &> 0, \ \forall \ x > 0 \end{aligned}$$

Therefore, f is strictly increasing for all $x > 0$

Therefore,

$$\begin{aligned} 0 < x &\Rightarrow f(0) < f(x) \\ &\Rightarrow \log(1+0) - \left(0 - \frac{0^2}{2}\right) < \log(1+x) - \left(x - \frac{x^2}{2}\right) \\ &\Rightarrow 0 < \log(1+x) - \left(x - \frac{x^2}{2}\right) \\ &\Rightarrow x - \frac{x^2}{2} < \log(1+x) - - - - - (i) \end{aligned}$$

Next, define, $g(x) = \left(x - \dfrac{x^2}{2(1+x)}\right) - \log(1+x)$

Here,

$$\begin{aligned} g'(x) &= 1 - \frac{2(1+x)(2x) - x^2(2)}{4(x+1)^2} - \frac{1}{1+x} \\ &= \frac{x}{1+x} - \frac{4x + 4x^2 - 2x^2}{4(x+1)^2} \\ &= \frac{x}{1+x} - \frac{2x + x^2}{2(x+1)^2} \\ &= \frac{2x(x+1) - (2x + x^2)}{2(x+1)^2} \\ &= \frac{x^2}{2(x+1)^2} \\ &> 0, \ \forall \ x > 0 \end{aligned}$$

Therefore, g is strictly increasing for all $x > 0$
Therefore,

$$\begin{aligned} 0 < x &\Rightarrow g(0) < g(x) \\ &\Rightarrow \left(0 - \frac{0^2}{2(0+1)}\right) - \log(1+0) < \left(x - \frac{x^2}{2(x+1)}\right) - \log(1+x) \\ &\Rightarrow 0 < \left(x - \frac{x^2}{2(x+1)}\right) - \log(1+x) \\ &\Rightarrow \log(1+x) < x - \frac{x^2}{2(x+1)} - - - - - - (ii) \end{aligned}$$

From (i)and (ii) it follows that $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, \ \forall x > 0$

12. **Prove that,** $\frac{x}{1+x} < \log(1+x) < x$ **for all** $x > 0$**.**

Solution:

Define $f(x) = \log(1+x) - \frac{x}{1+x}$
Here,

$$\begin{aligned} f'(x) &= \frac{1}{1+x} - \frac{(1+x)(1) - x(1)}{(x+1)^2} \\ &= \frac{1}{1+x} - \frac{1}{(x+1)^2} \\ &= \frac{(1+x) - 1}{(x+1)^2} \\ &= \frac{x}{(x+1)^2} \\ &> 0, \forall x > 0 \end{aligned}$$

Therefore, f is strictly increasing for all $x > 0$
Therefore,

$$\begin{aligned} 0 < x &\Rightarrow f(0) < f(x) \\ &\Rightarrow \log(1+0) - \frac{0}{1+0} < \log(1+x) - \frac{x}{1+x} \\ &\Rightarrow 0 < \log(1+x) - \frac{x}{1+x} \\ &\Rightarrow \frac{x}{1+x} < \log(1+x) ------ (i) \end{aligned}$$

Again define $g(x) = x - \log(1+x)$
Here,

$$\begin{aligned} g'(x) &= 1 - \frac{1}{x+1} \\ &= \frac{(x+1)-1}{x+1} \\ &= \frac{x}{x+1} \\ &> 0, \forall x > 0 \end{aligned}$$

Therefore, g is strictly increasing for all $0 < x$
Therefore,

$$\begin{aligned} 0 < x &\Rightarrow g(0) < g(x) \\ &\Rightarrow 0 - \log(1+0) < x - \log(1+x) \\ &\Rightarrow 0 < x - \log(1+x) \\ &\Rightarrow \log(x+1) < x ------ (ii) \end{aligned}$$

From (i)and (ii) it follows that $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$

13. State and prove the Darboux's theorem for derivable function.

Darboux's Theorem:
If a function f is derivable on a closed interval $[a, b]$ such that $f'(a)$ and $f'(b)$ are of opposite signs then there exists at least one point c between a and b such that $f'(c) = 0$

Proof:
Let us suppose $f'(a) < 0$ and $f'(b) > 0$

Since $f'(a) < 0$, the function f is decreasing at a in $[a, b]$

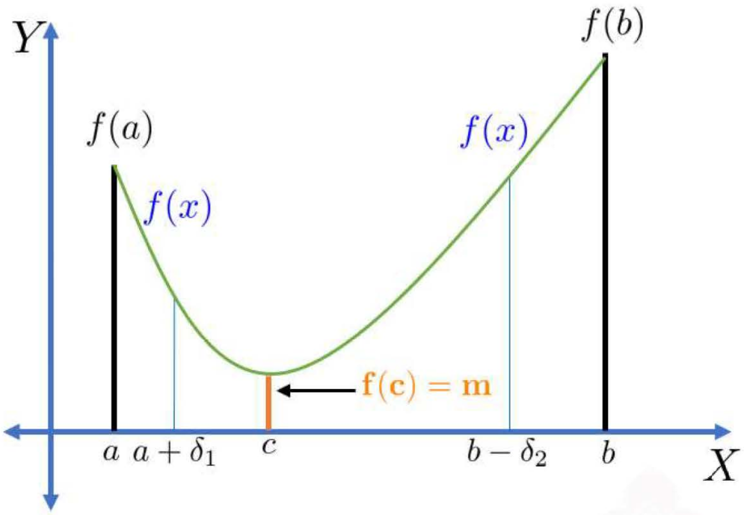
Therefore, there is some $\delta_1 > 0$ such that

$$f(a) > f(x), \ \forall \ x \in (a, a+\delta_1) \quad \text{--- (1)}$$

Also, as $0 < f'(b)$, the function f is increasing at b in $[a, b]$

Therefore, there is some $\delta_2 > 0$ such that

$$f(x) < f(b), \ \forall \ x \in (b-\delta_2, b) \quad \text{--- (2)}$$



$$f(x) \leqslant f(a), \ \forall \ x \in (a, a+\delta_1)$$

and

$$f(x) \leqslant f(b), \ \forall \ x \in (b-\delta_2, b)$$

Figure 5: f must have infimum at an interior point c

Now, as f is derivable on $[a, b]$ it is continuous on the interval. Since f is continous on the closed interval $[a, b]$ it is bounded and attains its bounds.

Suppose m is the infimum of f on $[a, b]$. Then there must be some c in $[a, b]$ such that

$$f(c) = m$$

As m is the infimum of f on $[a, b]$, from (1) and (2) it is follows that $f(a) \neq m$ and $f(b) \neq m$

But then, $f(a) \neq f(c)$ and $f(b) \neq f(c)$.

This implies that $a \neq c$ and $b \neq c$. Therefore

$$c \in (a, b)$$

Thus, c is an interior point of $[a, b]$.

Next we show that $f'(c) \nless 0$ and $f'(c) \ngtr 0$

If possible suppose, $f'(c) < 0$.

Then f is decreasing at c. Therefore there exists some $\delta_3 > 0$ such that

$$f(c) > f(x), \ \forall \ \ x \in (c, c+\delta_3)$$

But then we have,

$$m > f(x), \ \forall \ \ x \in (c, c+\delta_3)$$

which is not possible as m is the infimum of f in $[a, b]$. Therefore we must have

$$f'(c) \nless 0$$

Again, if possible suppose, $f'(c) > 0$.

Then f is increasing at c. Therefore there exists some $\delta_4 > 0$ such that

$$f(c) > f(x), \ \forall \ \ x \in (c-\delta_4, c)$$

But then we have,

$$m > f(x), \ \forall \ x \in (c - \delta_4, c)$$

which is not possible as m is the infimum of f in $[a, b]$. Therefore we must have

$$f'(c) \not> 0$$

As R is an ordered field, by the low of Trichotomy we must have,

$$f'(c) = 0$$

14. State and prove Rolle's theorem

Proof:

If a function f is continuous on $[a, b]$ then it is bounded on $[a, b]$ and attains its bounds at some points in $[a, b]$. If m and M are the infimum and the supremum of f in $[a, b]$ then for some points c and d in $[a, b]$ we have,

$$f(c) = m \quad \text{and} \quad f(d) = M$$

If $m = M$ then f is a constant function on $[a, b]$. In that case for every $c \in [a, b]$ we get $f(c) = 0$.

Now, if $m \neq M$ then any given number must be different from atleast one of m and M.

Therefore, we have either $f(a) \neq m$ or $f(a) \neq M$ Suppose, $f(a) \neq m$.

Therefore,

$$f(a) \neq m \Rightarrow f(a) \neq f(c) \Rightarrow a \neq c$$

and

$$f(b) \neq m \Rightarrow f(b) \neq f(c) \Rightarrow b \neq c$$

Hence

$$c \in (a, b)$$

Finally, we show that $f(c) \not< 0$ and $f(c) \not> 0$.

If possible, suppose $f'(c) < 0$. Then f is a decreasing function at c. Therefore there exists some $\delta_1 > 0$ such that

$$f(c) > f(x), \ \forall \ x \in (c, c + \delta_1)$$

But then

$$m > f(x), \ \forall \ x \in (c, c + \delta_1)$$

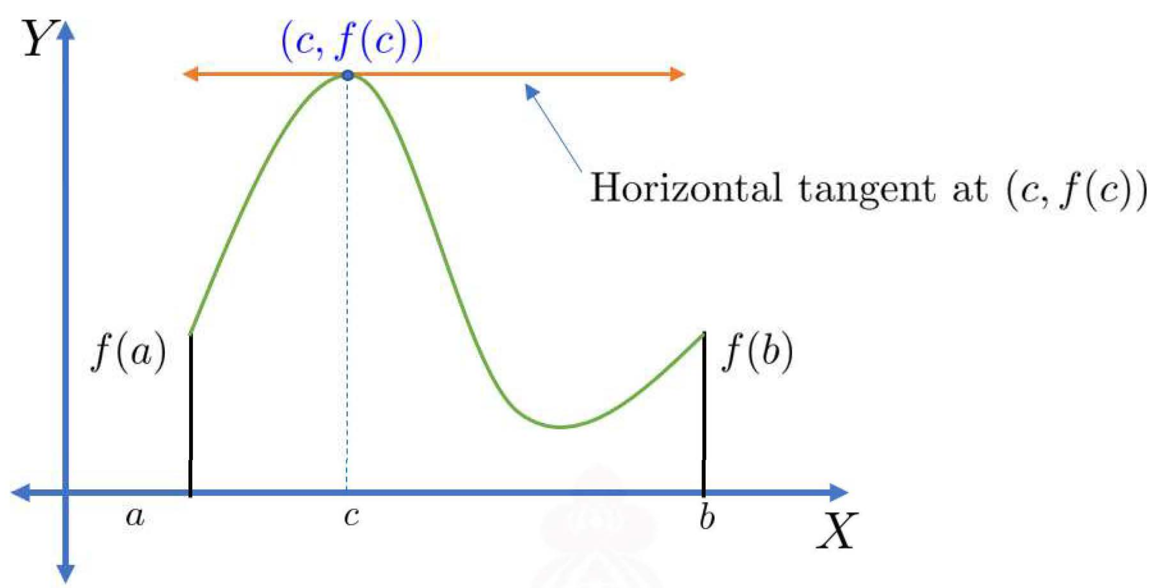
This is not possible as m is the infimum of f on $[a, b]$

Therefore

$$f(c) \not< 0$$

Again, if possible, suppose $f'(c) > 0$. Then f is an increasing function at c. Therefore there exists some $\delta_2 > 0$ such that

$$f(c) > f(x), \ \forall \ x \in (c - \delta_2, c)$$



But then

$$m > f(x), \quad \forall \ x \in (c-\delta_2, c)$$

This is not possible as m is the infimum of f on $[a, b]$

Therefore

$$f(c) \not> 0$$

As R is an ordered field, by the low of Trichotomy we must have,

$$f(c) = 0$$

15. Discuss Geometric Meaning of Rolle's theorem.

Geometric Interpretation of Rolle's Theorem

The Rolle's theorem states the following,

If a function f defined on $[a, b]$ is
(i) continuous on $[a, b]$
(ii) differentiable on (a, b) and
(iii) $f(a) = f(b)$

then there exists atleast one real number c between a and b such that $f'(c) = 0$

Geometrically, it can be said that if a function f is continuous on $[a, b]$ and derivable on (a, b) such that the ordinates $f(a)$ and $f(b)$ at the end points of $[a, b]$ are equal then there is atleast point $c \in (a, b)$ such that the the tangent at the point $(c, f(c))$ on the graph of $y = f(x)$ is parallet to the $X-$-axis. In other words the slope of the tangent at $(c, f(c))$ is

$$f'(c)0$$

16. Discuss Algebraic Meaning of Rolle's theorem.

Algebraic Interpretation of Rolle's Theorem
The Rolle's theorem states the following,

If a function f defined on $[a, b]$ is
(i) continuous on $[a, b]$
(ii) differentiable on (a, b) and
(iii) $f(a) = f(b)$

then there exists atleast one real number c between a and b such that $f'(c) = 0$

Algebraically, it can be said that there is atleast **ROOT** of the equation $f'(x) = 0$ in (a, b). In other words the equation $f'(x) = 0$ has atleast one zero in (a, b).

17. State and prove Lagrange's Mean Value theorem

Proof:
Define a function ϕ on $[a, b]$ as follows,

$$\phi(x) = f(x) + Ax$$

where A is to be determined such that $\phi(a) = \phi(b)$.
In that case, we must have,

$$\begin{aligned} f(a) + Aa &= f(b) + Ab \\ \therefore \quad A(a-b) &= f(b) - f(a) \\ \therefore \quad A &= -\frac{f(b) - f(a)}{b-a} \end{aligned}$$

Now $\phi(x)$ is a sum of two functions, namely $f(x)$ and Ax, which are continuous on $[a, b]$ and derivable on (a, b). Therefore, we have the following for $\phi(x)$,

(1) $\phi(x)$ is continuous on $[a, b]$
(2) $\phi(x)$ is derivable on (a, b)
(3) $\phi(a) = \phi(b)$

Therefore, by the Rolle's therorem there exists some $c \in (a, b)$ such that

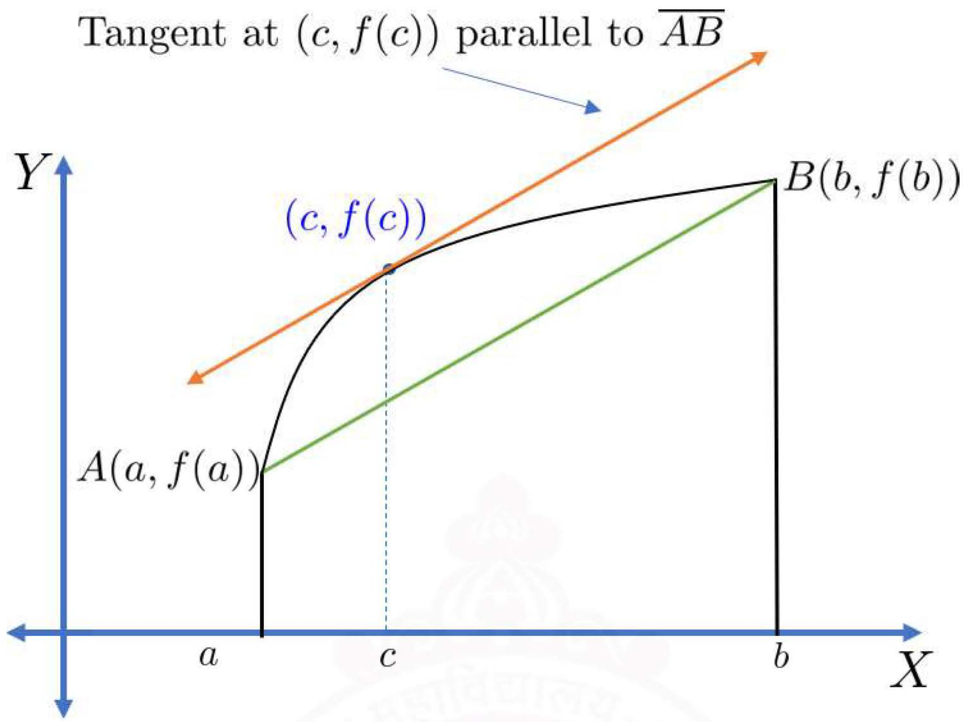
$$\phi'(c) = 0$$

Since

$$\phi'(x) = f'(x) + A$$

we get,

$$f'(c) + A = 0$$



$$\therefore\ f'(c) = -A$$

Hence,

$$\mathbf{f'(c) = \frac{f(b) - f(a)}{b - a}}$$

18. Discuss Geometric Meaning of Lagrange's theorem

Geometric Interpretation of Lagrange's Theorem

The Lagrange's theorem states the following,

(1) continuous on $[a, b]$ and
(2) differentiable on (a, b)
then there exists atleast one real number c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrically, it can be said that if a function f is continuous on $[a, b]$ and derivable on (a, b) then there is atleast one point $c \in (a, b)$ such that the the tangent at the point $(c, f(c))$ on the graph of $y = f(x)$ is parallel to the chord $\overline{AB}$ joining the points $A(a, f(a)$ and $B(b, f(b)$.

19. State and prove Cauchy's Mean Value theorem

Proof:

Define a function ϕ on $[a, b]$ as follows,

$$\phi(x) = f(x) + Ag(x)$$

where A is to be determined such that $\phi(a) = \phi(b)$.
In that case, we must have,

$$f(a) + Ag(a) = f(b) + Ag(b)$$
$$\therefore \quad A(g(a) - g(b)) = f(b) - f(a)$$
$$\therefore \quad A = -\frac{f(b) - f(a)}{g(b) - g(a)}$$

Now $\phi(x)$ is a sum of two functions, namely $f(x)$ and $Ag(x)$, which are continuous on $[a, b]$ and derivable on (a, b). Therefore, we have the following for $\phi(x)$,

(1) $\phi(x)$ is continuous on $[a, b]$
(2) $\phi(x)$ is derivable on (a, b)
(3) $\phi(a) = \phi(b)$

Therefore, by the Rolle's therorem there exists some $c \in (a, b)$ such that

$$\phi'(c) = 0$$

Since

$$\phi'(x) = f'(x) + Ag'(x)$$

we get,

$$f'(c) + Ag'(c) = 0$$
$$\therefore \quad f'(c) = -Ag'(c)$$
$$\therefore \quad \frac{f'(c)}{g'(c)} = -A$$

Hence,

$$\boxed{\mathbf{\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}}}$$

20. If a function $f(x)$ satisfies the conditions of the Lagrange's Mean Value Theorem and $f'(x) = 0, \ \forall x \in [a, b]$ then prove that f is constant on $[a, b]$

Proof:
Function $f(x)$ satisfies the conditions of the Lagrange's Mean Value Theorem on $[a, b]$.

Now, for any $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ we have

$$[x_1, x_2] \subset [a, b]$$

Therefore $f(x)$ also satisfies the conditions of Lagrange's Mean Value Theorem on $[x_1, x_2]$.

Therefore, there exists some $c \in (x_1, x_2)$ such that,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{- - - (1)}$$

As it is given that $f'(x) = 0,\ \forall x \in [a, b]$, we have $f'(c) = 0$.
Therefore, from (1) we get,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$$
$$\therefore \quad f(x_2) - f(x_1) = 0$$
$$\therefore \quad f(x_2) = f(x_1)$$

As choice of $x_1, x_2 \in [a, b]$ is arbitrary, it follows that f assumes same value for all $x \in [a, b]$.

Hence, f is constant on $[a, b]$.

21. If two functions have equal derivatives at all points then show that they differ only by a constant

Proof:
Let f and g be two functions defined on (a, b) such that

$$f'(x) = g'(x) \quad \forall x \in (a, b)$$

Now define,

$$h(x) = f(x) - g(x), \quad \forall x \in (a, b)$$

As f and g both are derivable on (a, b), h is also derivable on (a, b) and

$$h'(x) = f'(x) - g'(x) \quad \forall x \in (a, b)$$

$$h'(x) = 0 \quad \forall x \in (a, b)$$

Hence h is a constant function on (a, b).

Therefore, for some constant k ,

$$h(x) = k, \quad \forall x \in (a, b)$$

$$\therefore \quad f(x) - g(x) = k, \quad \forall x \in (a, b)$$

Hence, $f(x)$ and $g(x)$ differ only by a constant.

22. If f is continuous on $[a, b]$, derivable on (a, b) and $f'(x) > 0,\ \forall x \in (a, b)$ then prove that f is strictly incerasing function on $[a, b]$

Proof:

Here, function f is continuous on $[a, b]$ and derivable on (a, b).

Therefore f stisfies all the conditions for Lagrange's Mean Value Theorem.

Moreover it is given that $f'(x) > 0, \quad \forall x \in (a, b)$

Now, for any $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ we have

$$[x_1, x_2] \subset [a, b]$$

Therefore $f(x)$ also satisfies the conditions of Lagrange's Mean Value Theorem on $[x_1, x_2]$.

Therefore, there exists some $c \in (x_1, x_2)$ such that,

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \quad \text{- - - (1)}$$

As it is given that $f'(x) > 0, \ \forall x \in [a, b]$, we have $f'(c) > 0$.
Therefore, from (1) we get,

$$0 < \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$
$$\therefore \quad 0 < f(x_2) - f(x_1)$$
$$\therefore \quad f(x_1) < f(x_2)$$

Thus we have,

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2), \quad \text{for} \quad x_1, x_2 \in [a, b]$$

Hence, f is strictly increasing on $[a, b]$.

23. Show that $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$, for some θ where $0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$

Solution:
For $0 < \alpha < \beta < \frac{\pi}{2}$, define,

$$f(x) = \sin x \text{ and } g(x) = \cos x \text{ for } x \in [\alpha, \beta]$$

As sin and cos both are continuous on $[0, \frac{\pi}{2}]$ and derivable on $(0, \frac{\pi}{2})$ they are also continuous on $[\alpha, \beta]$ and derivable on (α, β).

Also, $g'(x) = -\sin x \neq 0, \ \forall x \in (\alpha, \beta)$.

So, the Cauchy's Mean Value theorem is appicable. Therefore there is some $\theta \in (\alpha, \beta)$ such that,

$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)}$$
$$\therefore \quad \frac{\cos\theta}{-\sin\theta} = \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha}$$

$$\therefore \ \frac{\cos\theta}{\sin\theta} = \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha}$$

$$\therefore \ \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta$$

24. **A twice differentiable function f is such that $f(a) = f(b) = 0$ and $f(c) > 0$ for $a < c < b$. Prove that there is al least one value ξ between a and b for which $f''(\xi) < 0$.**

Solution:

Here, f is a twice differentiable function on (a, b) such that $f(a) = f(b) = 0$

Therefore, f'' exists on (a, b), hence f' also exists on (a, b).

Also, at a point $c \in (a, b)$ it is given that $f(c) > 0$.

Applying Lagrange's Mean Value Theorem on $[a, c]$ and $[c, b]$ we get some $\xi_1 \in (a, c)$ and $\xi_2 \in (c, b)$ such that

$$f'(\xi_1) = \frac{f(c) - f(a)}{c - a} \quad \text{and} \quad f'(\xi_2) = \frac{f(b) - f(c)}{b - c}$$

As, $f(a) = f(b) = 0$, we get

$$f'(\xi_1) = \frac{f(c)}{c - a} \quad \text{and} \quad f'(\xi_2) = -\frac{f(c)}{b - c} \quad \text{--- (1)}$$

Also, $f'(x)$ is continuous on $[\xi_1, \xi_2]$. Therefore, again applying Lagrange's Mean Value Theorem to $f'(x)$ on $[\xi_1, \xi_2]$ we get some $\xi \in [\xi_1, \xi_2]$ such that,

$$f''(\xi) = \frac{f'(\xi_2) - f(\xi_1)}{\xi_2 - \xi_1}$$

Substituting for $f'(\xi_1)$ and $f'(\xi_2)$ from (1), we get,

$$\begin{aligned} f''(\xi) &= \frac{-\frac{f(c)}{b-c} - \frac{f(c)}{c-a}}{\xi_2 - \xi_1} \\ &= -\frac{f(c)}{\xi_2 - \xi_1}\left[\frac{1}{b-c} + \frac{1}{c-a}\right] \\ &= -\frac{f(c)}{\xi_2 - \xi_1}\left[\frac{(c-a) + (b-c)}{(b-c)(c-a)}\right] \\ &= -\frac{f(c)}{\xi_2 - \xi_1}\left[\frac{b-a}{(b-c)(c-a)}\right] \end{aligned}$$

Since, $f(c) > 0$ and all the numbers in each of the brackets on the RHS are positive, we have

$$f''(\xi) < 0$$

25. **State and prove Taylor's theorem.**

Taylor's Theorem:

If a function f defined on $[a, a+h]$ is such that
(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$ and
(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$, then there exists atleast one real number θ between 0 and 1 such that

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^{(n)}(a+\theta h)$$

where p is a possitive integer.

Proof:

As $f^{(n-1)}$ is continuous on $[a, a+h]$, it implies that

$$f, f', f'', \ldots, f^{(n-1)} \text{ all exist and are continuous on } [a, a+h]$$

Define,

$$\phi(x) = f(x) + \frac{((a+h)-x)}{1!}f'(x) + \frac{((a+h)-x)^2}{2!}f''(x) + \frac{((a+h)-x)^3}{3!}f'''(x) + \cdots + \frac{((a+h)-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + A(a+h-x)^p$$

Where A is a constant to be determined such that

$$\phi(a+h) = \phi(a)$$

For this we must have,

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + Ah^p \text{ - - - } (1)$$

Now, $f, f', f'', \ldots, f^{(n-1)}$ are continuous on $[a, a+h]$ implies that

$$\phi(x) \text{ is continuous on } [a, a+h] \text{ - - - } (2)$$

Moreover, $f^{(n)}$ exists on $(a, a+h)$ implies that $f, f', f'', \ldots, f^{(n-1)}$ are derivable on $(a, a+h)$. Therefore,

$$\phi'(x) \text{ is derivable on } (a, a+h) \text{ - - - } (3)$$

Also,

$$\phi(a+h) = \phi(a) \text{ - - - } (4)$$

From (2),(3) and (4) it follows that $\phi(x)$ satisfies all the conditions of Rolle's theorem and hence there exists some real number $\theta \in (0,1)$ such that

$$\phi'(a+\theta h) = 0$$

We have,

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!} f^{(n)}(x) - Ap(a+h-x)^{p-1}$$

Therefore, $\phi'(a+\theta h) = 0$ implies that

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{n}(a+\theta h) - Ap[a+h-(a+\theta h)]^{p-1} = 0$$

$$\therefore\ \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - Aph^{p-1}(1-\theta)^{p-1} = 0$$

$$\therefore\ \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - Aph^{p-1}(1-\theta)^{p-1} = 0$$

$$\therefore\ Aph^{p-1}((1-\theta)^{p-1}) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

$$\therefore\ A = \frac{h^{n-p}(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$$

Substituting for A in (1), we get

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$$

, where $\theta \in (0,1)$ and p is a postitive number.

26. Forms of remainders in Taylor's theorem.

(1) **Schlömilch and Röche form of remainder**

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$$

(2) **Cauchy's form of remainder**

$$R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

which can be obtained by taking $p=1$ in Schlömilch and Röche form of remainder.

(3) **Lagrange's form of remainder**

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

which can be obtained by taking $p=n$ in Schlömilch and Röche form of remainder.

27. **Prove Taylor's theorem with Cauchy's form of remainder by taking the function**

$$\phi(x) = f(x) + \frac{(a+h-x)}{1!}f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \cdots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + A(a+h-x)$$

Proof:
Let f be a function defined on $[a, a+h]$ is such that
(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$ and
(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$

As $f^{(n-1)}$ is continuous on $[a, a+h]$, it implies that

$$f, f', f'', \ldots, f^{(n-1)} \text{ all exist and are continuous on } [a, a+h]$$

Define,

$$\phi(x) = f(x) + \frac{((a+h)-x)}{1!}f'(x) + \frac{((a+h)-x)^2}{2!}f''(x) + \frac{((a+h)-x)^3}{3!}f'''(x) + \cdots + \frac{((a+h)-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + A(a+h-x)$$

Where A is a constant to be determined such that

$$\phi(a+h) = \phi(a)$$

For this we must have,

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + Ah \text{ - - - } (1)$$

Now, $f, f', f'', \ldots, f^{(n-1)}$ are continuous on $[a, a+h]$ implies that

$$\phi(x) \text{ is continuous on } [a, a+h] \text{ - - - } (2)$$

Moreover, $f^{(n)}$ exists on $(a, a+h)$ implies that $f, f', f'', \ldots, f^{(n-1)}$ are derivable on $(a, a+h)$. Therefore,

$$\phi'(x) \text{ is derivable on } (a, a+h) \text{ - - - } (3)$$

Also,

$$\phi(a+h) = \phi(a) \text{ - - - } (4)$$

From (2),(3) and (4) it follows that $\phi(x)$ satisfies all the conditions of Rolle's theorem and hence there exists some real number $\theta \in (0,1)$ such that

$$\phi'(a+\theta h) = 0$$

We have,

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - A$$

Therefore, $\phi'(a+\theta h) = 0$ implies that

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - A = 0$$

$$\therefore \; \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h) - A = 0$$

$$\therefore \; A = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(a+\theta h)$$

Substituting for A in (1), we get

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(}n)(a+\theta h)$$

Where the last term in the expansion is the Cauchy's form of Remainder.

> 28. **State and prove Generalised Mean Value theorem.**
>
> **OR**
>
> **Deduce Taylor's theorem from Mean Value Theorem.**

Generalised Mean Value theorem:

Let f be a function defined on $[a, a+h]$ such that

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$ and

(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$,

then there exists atleast one real number θ between 0 and 1 such that

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h)$$

Proof:

As $f^{(n-1)}$ is continuous on $[a, a+h]$, it implies that

$$f, f', f'', \ldots, f^{(n-1)} \text{ all exist and are continuous on } [a, a+h]$$

Define,

$$\phi(x) = f(x) + \frac{((a+h)-x)}{1!}f'(x) + \frac{((a+h)-x)^2}{2!}f''(x) + \frac{((a+h)-x)^3}{3!}f'''(x) + \cdots + \frac{((a+h)-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$

As $f, f', f'', \ldots, f^{(n-1)}$ are continuous on $[a, a+h]$ implies that

$$\phi(x) \text{ is continuous on } [a, a+h] \text{ - - - (1)}$$

Moreover, $f^{(n)}$ exists on $(a, a+h)$ implies that $f, f', f'', \ldots, f^{(n-1)}$ are derivable on $(a, a+h)$. Therefore,

$$\phi'(x) \text{ is derivable on } (a, a+h) \text{ - - - (2)}$$

Therefore by Lagrange's Mean Value Theorem there exists some $\theta \in (0,1)$ such that

$$\phi(a+h) = \phi(a) + h\phi'(a+\theta h)$$

Therefore,

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + h\phi'(a+\theta h) \text{ - - - (3)}$$

As

$$\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x)$$

we have,

$$\phi'(a+\theta h) = \frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}f^{(n)}(a+\theta h)$$

$$\therefore \ \phi'(a+\theta h) = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h)$$

Substituting in (3) we get,

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h)$$

29. State Maclaurin's theorem and deduce it from Taylor's theorem.

Maclaurin's theorem:

Let f be a function defined on $[0, h]$ such that
(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[0, h]$ and
(ii) the n^{th} derivative $f^{(n)}$ exists on $(0, h)$,
then for each $x \in (0, h)$ there exists atleast one real number $\theta \in (0, 1)$ such that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x)$$

Proof:

As $f^{(n-1)}$ is continuous on $[0, h]$ and $f^{(n)}$ exists on $(0, h)$ for any $x \in (0, h)$ it implies that $f^{(n-1)}$

is continuous on $[0, x]$ and $f^{(n)}$ exists on $(0, x)$.

Therefore, by Taylor's theorem, for a postive integer p there exists some $\theta \in (0, 1)$ such that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^{(n)}(\theta x)$$

30. Forms of remainders in Maclaurin's theorem.

(1) **Schlömilch and Röche form of remainder**

$$R_n = \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^{(n)}(\theta x)$$

(2) **Cauchy's form of remainder**

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x)$$

which can be obtained by taking $p = 1$ in Schlömilch and Röche form of remainder.

(3) **Lagrange's form of remainder**

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x)$$

which can be obtained by taking $p = n$ in Schlömilch and Röche form of remainder.

31. Taylor's Series

Taylor's Series

For a function f defined on $[a, a+h]$ if
(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$ and
(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$

then by Taylor's theorem there is some $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots\cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + R_n$$

Where R_n is the Schlömilch and Röche form or Cauchy's from or Lagrange's form of raminder.

If we take

$$S_n == f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

then we can write,

$$f(a+h) = S_n + R_n \quad \text{- - - (1)}$$

Suppose f possesses derivative of every order on $(a, a+h)$ and $\lim_{n\to\infty} R_n = 0$. In that case

$$\lim_{n\to\infty} f(a+h) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n$$

Therefore, we get

$$f(a+h) = f(a) + \frac{h}{1!}f'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots$$

The series on the right hand side is called **Taylor's series** for $f(x)$.

By taking $a + h = x$ we can express the series in the following form

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \cdots$$

which the **Power Series** expansion of $f(x)$ in powers of $(x - a)$.

32. Maclaurin's Series

Maclaurin Series

For a function f defined on $[0, h]$ if

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[0, h]$ and

(ii) the n^{th} derivative $f^{(n)}$ exists on $(0, h)$

then for every $x \in (0, h)$, by Maclaurin's theorem, there is some $\theta \in (0, 1)$ such that

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

Where R_n is the Schlömilch and Röche form or Cauchy's from or Lagrange's form of raminder.

If we take

$$S_n = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)$$

then we can write,

$$f(x) = S_n + R_n \quad \text{- - - (1)}$$

Suppose f possesses derivative of every order on $(0, h)$ and $\lim_{n\to\infty} R_n = 0$. In that case

$$\lim_{n\to\infty} f(x) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} R_n$$

Therefore, we get

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

The series on the right hand side is called **Maclaurin's series** for $f(x)$. which the **Power Series** expansion of $f(x)$ in powers of x.

33. Obtain series expansion of e^x.

<u>**Answer:**</u>
Function $f(x) = e^x$ possesses derivatives of every order for every $x \in R$ and $f^{(n)}(x) = e^x$, $\forall n$. Therefore, the Maclaurin's expansion with remainder R_n is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

In the expansion, if we consider R_n to be the Lagrange's form of remainder then for $\theta \in (0,1)$,

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n}{n!}e^{\theta x}$$

Here,

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{x^n}{n!}e^{\theta x} \\ &= \left(\lim_{n\to\infty} \frac{x^n}{n!}\right) e^{\theta x} \\ &= 0 \end{aligned}$$

Since $\lim_{n\to\infty} R_n = 0$, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied. Now,

$$f(0) = f'(0) = f''(0) = f'''(0) = \cdots = f^{(n)}(0) = \cdots = e^0 = 1$$

As the general form of Maclaurin's expansion is given by,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(a) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

We get the following Maclaurin's infinite series for $f(x) = e^x$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

34. Obtain series expansion of $\cos x$.

<u>**Answer:**</u>
Function $f(x) = \cos x$ possesses derivatives of every order for every $x \in R$ and

$$f^{(n)}(x) = \cos\left(\frac{n\pi}{2} + x\right)$$

.Therefore, the Maclaurin's expansion with remainder R_n is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

In the expansion, if we consider R_n to be Lagrange's form of remainder then for $\theta \in (0,1)$

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right)$$

Therefore

$$|R_n| = \left|\frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$\therefore |R_n| = \left|\frac{x^n}{n!}\right| \left|\cos\left(\frac{n\pi}{2} + \theta x\right)\right|$$

Since,

$$\left|\cos\left(\frac{n\pi}{2} + \theta x\right)\right| \leqslant 1$$

we get,

$$\therefore |R_n| \leqslant \left|\frac{x^n}{n!}\right|$$

As,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

we conclude that,

$$\lim_{n\to\infty} R_n = 0$$

Therefore, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied. Now,

$$\begin{aligned} f(0) &= \cos 0 = 1 \\ f'(0) &= -\sin 0 = 0 \\ f''(0) &= -\cos 0 = -1 \\ f^{(iv)}(0) &= \sin 0 = 0 \\ f^{(v)}(0) &= \cos 0 = 1 \\ &\dots \end{aligned}$$

As the general form of Maclaurin's expansion is given by,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

We get the following Maclaurin's infinite series for $f(x) = \cos x$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

35. **Obtain series expansion of $\log(1+x)$ for $-1 < x \leqslant 1$.**

Answer:

For the function $f(x) = \log(1+x)$ we have, $f^{(n)}(x) = \dfrac{(-1)^{(n-1)}(n-1)!}{(1+x)^n}$

Hence, $f(x)$ possesses derivatives of every order for every $-1 < x \leqslant 1$ and they are continuous for $|x| < 1$. Therefore, the Maclaurin's expansion with remainder R_n is given by

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

In the expansion, if we consider R_n to be Lagrange's form of remainder then for $\theta \in (0,1)$

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x)$$

$$\therefore \; R_n = \frac{x^n}{n!}\frac{(-1)^{(n-1)}(n-1)!}{(1+\theta x)^n} = (-1)^{(n-1)}\frac{1}{n}\left(\frac{x}{1+\theta x}\right)^n$$

We consider the cases $0 \leqslant x \leqslant 1$ and $-1 < x < 0$ seperately.

<u>when</u> $\mathbf{0 \leqslant x \leqslant 1}$

As $0 < \theta < 1$ it is clear that $x \leqslant 1 < 1 + \theta x$. Hence

$$0 < \frac{x}{1+\theta x} < 1$$

Therefore,

$$\lim_{n\to\infty}\left(\frac{x}{1+\theta x}\right)^n = 0$$

Moreover,

$$\lim_{n\to\infty}\frac{1}{n} = 0$$

Therefore, for $0 \leqslant x \leqslant 1$ we have

$$\lim_{n\to\infty} R_n = 0$$

Hence, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied for $0 \leqslant x \leqslant 1$.

<u>when</u> $\mathbf{-1 < x < 0}$

In this case x may or may not be less than $1 + \theta x$. Hence nothing can be predicated about $\lim\limits_{n\to\infty}\left(\dfrac{x}{1+\theta x}\right)$.

So, with Lagrange's form of remainder no conclusion is possible regarding infinite series.

Next, let us consider, Cauchy's form of remainder given by

$$R_n = \frac{x^n(1-\theta)^{(n-1)}}{(n-1)!}f^{(n)}(\theta x)$$

Therefore,

$$R_n = \frac{x^n(1-\theta)^{(n-1)}}{(n-1)!}\frac{(-1)^{(n-1)}(n-1)!}{(1+x)^n} = (-1)^{(n-1)}x^n\left(\frac{(1-\theta)}{(1+\theta x)}\right)^n\frac{1}{(1+\theta x)}$$

As $1-\theta < 1+\theta x$ we have

$$\lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{(n-1)} = 0$$

Moreover,

$$\lim_{n\to\infty} x^n = 0 \quad \text{and} \quad \frac{1}{(1+\theta x)} < \frac{1}{(1-|x|)}$$

Therefore, for $-1 < x < 0$ we have

$$\lim_{n\to\infty} R_n = 0$$

Hence, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied for $-1 < x < 0$ also. Now,

$$f(0) = \log(0+1) = 0$$

and

$$f^{(n)}(0) = \frac{(-1)^{(n-1)}(n-1)!}{(1+0)^n} = (-1)^{(n-1)}(n-1)!$$

As the general form of Maclaurin's expansion is given by,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

We get the following Maclaurin's infinite series for $f(x) = \log(1+x)$,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

for $-1 < x \leqslant 1$.

36. Obtain series expansion of $(1+x)^m$.

Answer:

For the function $f(x) = (1+x)^m$ we shall consider two cases depending on whether m is a positive integer or not.

Case:1 m is a positive integer :

In this case, for every $x \in R$ and for each $m \leqslant n$ we have

$$f^{(n)}(x) = m(m-1)(m-2)\cdots(m-n+1)(1+x)^{m-n}$$

Hence $f(x)$ possesses continuous derivatives of all orders upto m.

Moreover, for $m < n$ we have, $f^{(n)} = 0$. This implies that

$$\lim_{n\to\infty} R_n = 0$$

Hence, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied for $\forall x \in R$, when m is a positive integer.

We have, $f(0) = 1$ and $f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)$.

Therefore, we get the following Maclaurin's infinite series for $f(x) = (1+x)^m$, $\forall x \in R$ and postitive integer m

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots + x^m$$

Case:2 m is a non-positive integer :
In this case, for every $x \neq -1$ function $f(x)$ possesses continuous derivatives of all orders.

Now, we shall consider the cases of $|x| < 1$ and $|x| > 1$ seperately.

For $-1 < x < 1$, (i.e. $|x| < 1$) let us consider Cauchy's form of remainder in Maclaurin's expansion.

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$$

$$\therefore R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!} m(m-1)(m-2)\cdots(m-n+1)(1+\theta x)^{m-n}$$

$$\therefore R_n = \left(\frac{m(m-1)(m-2)\cdots(m-n+1)x^n}{(n-1)!}\right)\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1} \quad \text{--- (1)}$$

Now, for $|x| < 1$ we have,

$$\lim_{n\to\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^n = 0$$

Since $1-\theta < 1+\theta x$ we have $\dfrac{1-\theta}{1+\theta x}$. Hence,

$$\lim_{n\to\infty}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1} = 0$$

Also, as $0 < \theta < 1$, for $m > 1$ we have,

$$(1+\theta x)^{m-1} < (1+|\theta x|)^{m-1} < (1+|x|)^{m-1}$$

and for $m < 1$ we have,

$$(1+\theta x)^{m-1} = \frac{1}{(1+\theta x)^{1-m}} < \frac{1}{(1-|x|)^{1-m}}$$

Hence, from (1) it follows that

$$\lim_{n\to\infty} R_n = 0 \quad \text{for} \quad |x| < 1$$

Hence, the condition for Maclaurin's infinite expansion for $f(x)$ is satisfied for $|x| < 1$ and non-positive integer m

The general form of Maclaurin's expansion is given by,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(a) + \cdots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \cdots$$

We have, $f(0) = 1$ and $f^{(n)}(0) = m(m-1)(m-2)\cdots(m-n+1)$

Therefore, we get the following Maclaurin's infinite series for $f(x) = (1+x)^m$, $\forall |x| < 1$ and non-postitive integers m

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$$

Finally let us consider the case when $|x| > 1$. In that case,

$$\lim_{n\to\infty} \frac{m(m-1)(m-2)\cdots(m-n+1)}{(n-1)!}x^n \neq 0$$

Therefore,

$$\lim_{n\to\infty} R_n \neq 0 \quad \text{for} \quad |x| > 1$$

Hence, for $|x| > 1$ the Maclaurin's expansion is not possible.