T.Y.B.Sc. : Semester - V

US05CMTH22(T)

Theory Of Real Functions

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US05CMTH22(T)- UNIT : III

1. Explicit and Implicit FunctionS.

Explicit and Implicit Function

If $x_1, x_2, \ldots, x_n$ are independent variables and u is a dependent variable which has its dependence on these variables by an explicit relation

$$u = f(x_1, x_2, \ldots, x_n)$$

then u is called an explicit function of $x_1, x_2, \ldots, x_n$.

In case a relation involving several variables is expressed by a relation like

$$\phi(x_1, x_2, \ldots, x_n) = 0$$

in which no single variable is expressed in terms of rest of the variables then it is called an implicit function.

2. Neighbourhoof of a point in R^2.

Neighnourhood of a point in R^2

For a point (a, b) and $\delta > 0$ a neighbourhood of (a, b) is defined as the set

$$\{(x, y) \in R^2 / \ |x - a < \delta, \ |y - b| < \delta\}$$

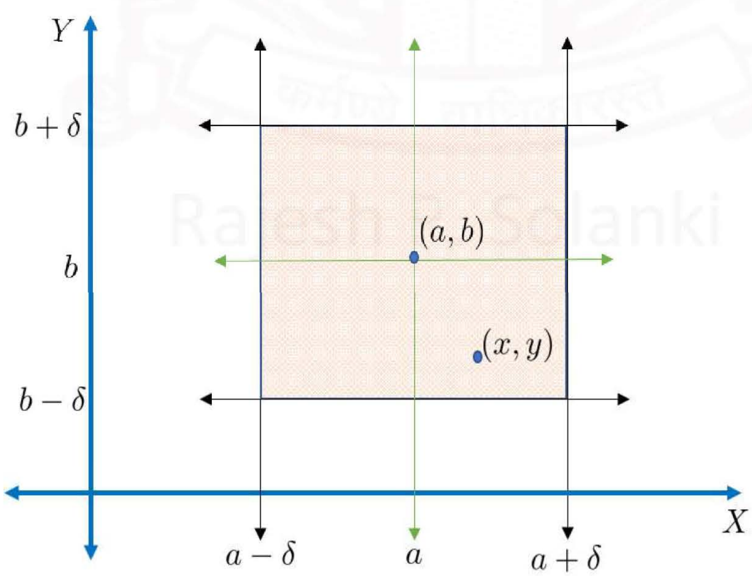

Figure 1: Neighbourhood of a point (a, b)

3. Limit Point

Limit Point
A point (ξ, η) is called a limit point of a subset S of R^2 if each neighbourhood of (ξ, η) contains infinitely many points of S.

4. Limit of a Function

Limit of a Function
Let $f(x, y)$ be a real valued funciton defined in some domain containing a deleted neighnourhood of a point (a, b) and l be a fixed real number. If for each $\epsilon > 0$ there exists some $\delta > 0$ such that

$$|f(x, y) - l| < \epsilon, \quad \text{whenever} \quad 0 < |x - a| < \delta, \ 0 < |y - b| < \delta$$

then l is said to be the the limit of $f(x, y)$ as (x, y) tends to (a, b) and it is written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = l$$

The limit is also known as **Double limit** or **simultaneous limit** .

5. Non-existence of limit

Non-existence of limit
From the definition of the limit of a funcion of two variables it follows that if

$$\lim_{(x,y)\to(a,b)} f(x, y) = l$$

then irrespective of the path we choose for (x, y) to approach (a, b) we must get the same value l, provided the limit through the path exists.

This implies that, if there exist atleast two paths say $y = \phi_1(x)$ and $y = \phi_2(x)$ such that the limits $\lim\limits_{(x,y)\to(a,b)} f(x, \phi_1(x))$ and $\lim\limits_{(x,y)\to(a,b)} f(x, \phi_2(x))$ both exist but

$$\lim_{(x,y)\to(a,b)} f(x, \phi_1(x)) \neq \lim_{(x,y)\to(a,b)} f(x, \phi_2(x))$$

then the **simultaneous limit does not exist.**

6. Repeated limits

Repeated limits
Let $f(x, y)$ be a function defined in some neighbourhood of a point (a, b). By keeping y constant in $f(x, y)$ it is a function of one variable x only and with that if the limit $\lim\limits_{(x\to a} f(x, y)$ exists then it is a function of y only, say $\phi(y)$. Then if the limit $\lim\limits_{y\to b} \phi(y)$ exists and it is equal to some real number λ then it is called the **Repeated Limit** of $f(x, y)$ as $x \to a$ and $y \to b$ and is written as

$$\lim_{y\to b} \lim_{x\to a} f(x, y) = \lambda$$

Similarly if the orders of limits is interchanged and the limit exists and it is equal to some real number λ' then it is called **Repeated Limit** of $f(x,y)$ as $y \to b$ and $x \to a$. This is written as

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lambda'$$

> 7. **For the function** $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \textbf{if} \quad x^2+y^2 \neq 0 \\ 0 & \textbf{if} \quad x+y=0 \end{cases}$
>
> **prove that** $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ **does not exist.**

Proof:
For a non-zero constant m let us evaluate the limit through the path $y = mx^2$

$$\begin{aligned} \lim_{x\to 0} f(x, mx^2) &= \lim_{x\to 0} \frac{x^2(mx^2)}{x^4+m^2x^4} \\ &= \lim_{x\to 0} \frac{mx^4}{x^4(1+m^2)} \\ &= \lim_{x\to 0} \frac{m}{1+m^2} \\ \therefore \lim_{(x,y)\to(0,0)} f(x,mx^2) &= \frac{m}{1+m^2} \end{aligned}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x^2$ and $y = m_2x^2$ then we get

$$\lim_{(x,y)\to(0,0)} f(x,m_1x^2) = \frac{m_1}{1+m_1^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x,m_2x^2) = \frac{m_2}{1+m_2^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x,m_1x^2) \neq \lim_{(x,y)\to(0,0)} f(x,m_2x^2)$$

Hence, the simultaneous limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

> 8. **Show that** $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$ **does not exist.**

Proof:

For a non-zero constant m let us evaluate the limit through the path $x = my^3$

$$\lim_{y\to 0} f(my^3, y) = \lim_{x\to 0} \frac{(my^3)y^3}{(my^3)^2 + y^6}$$
$$= \lim_{x\to 0} \frac{my^6}{y^6(m^2+1)}$$
$$= \lim_{x\to 0} \frac{m}{m^2+1}$$
$$\therefore \lim_{(x,y)\to(0,0)} f(my^3, y) = \frac{m}{m^2+1}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x^2$ and $y = m_2x^2$ then we get

$$\lim_{(x,y)\to(0,0)} f(m_1y^3, y) = \frac{m_1}{m_1^2+1} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(m_2y^3, y) = \frac{m_2}{m_2^2+1}$$

But then

$$\lim_{(x,y)\to(0,0)} f(m_1y^3, y) \neq \lim_{(x,y)\to(0,0)} f(m_2y^3, y)$$

Hence, the simultaneous limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

9. **Show that** $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ **does not exist.**

Proof:
For a non-zero constant m let us evaluate the limit through the path $y = mx$

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{2x(mx)}{x^2 + m^2x^2}$$
$$= \lim_{x\to 0} \frac{2mx^2}{x^2(1+m^2)}$$
$$= \lim_{x\to 0} \frac{2m}{1+m^2}$$
$$\therefore \lim_{(x,y)\to(0,0)} f(x, mx) = \frac{2m}{1+m^2}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x$ and $y = m_2x$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) = \frac{2m_1}{1+m_1^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, m_2x) = \frac{2m_2}{1+m_2^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) \neq \lim_{(x,y)\to(0,0)} f(x, m_2x)$$

Hence, the simultaneous limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist.

10. **Show that** $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2y^2 + (x^2-y^2)^2}$ **does not exist.**

Proof:
For a non-zero constant m let us evaluate the limit through the path $y = mx$

$$\begin{aligned}\lim_{x\to 0} f(x, mx) &= \lim_{x\to 0} \frac{x^2(mx)^2}{x^2(mx)^2 + (x^2-(mx)^2)^2} \\ &= \lim_{x\to 0} \frac{m^2x^4}{x^4m^2 + x^4(1-m^2)^2} \\ &= \lim_{x\to 0} \frac{m^2}{m^2+(1-m^2)^2} \\ \therefore \lim_{(x,y)\to(0,0)} f(x, mx) &= \frac{m^2}{m^2+(1-m^2)^2}\end{aligned}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x$ and $y = m_2x$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) = \frac{m_1^2}{m_1^2+(1-m_1^2)^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, m_2x) = \frac{m_2^2}{m_2^2+(1-m_2^2)^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) \neq \lim_{(x,y)\to(0,0)} f(x, m_2x)$$

Hence, the simultaneous limit $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

11. **Show that** $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x-y}$ **does not exist.**

Proof:
For a non-zero constant m let us evaluate the limit through the path $y = x - mx^3$

$$\begin{aligned}\lim_{x\to 0} f(x, x-mx^3) &= \lim_{x\to 0} \frac{x^3+(x-mx^3)^3}{x-(x-mx^3)} \\ &= \lim_{x\to 0} \frac{x^3+x^3(1-mx^2)^3}{mx^3} \\ &= \lim_{x\to 0} \frac{1+(1-mx^2)^3}{m} \\ &= \lim_{x\to 0} \frac{1+(1-3mx^2+3m^2x^4-m^3x^6)}{m} \\ &= \lim_{x\to 0} \frac{2-3mx^2+3m^2x^4-m^3x^6}{m} \\ &= \frac{2}{m} \\ \therefore \lim_{(x,y)\to(0,0)} f(x, x-mx^3) &= \frac{2}{m}\end{aligned}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = x - m_1x^3$ and $y = x - m_2x^3$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, x-m_1x^3) = \frac{2}{m_1} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, x-m_2x^3) = \frac{2}{m_2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, x - m_1x^3) \neq \lim_{(x,y)\to(0,0)} f(x, x - m_2x^3)$$

Hence, the simultaneous limit $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist.

12. **Show that** $\lim\limits_{(x,y)\to(0,0)} xy\dfrac{x^2 - y^2}{x^2 + y^2} = 0.$

Proof:

For any point (x, y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$xy\frac{x^2 - y^2}{x^2 + y^2} = (r\cos\theta)(r\sin\theta)\left[\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right]$$

$$= r^2\cos\theta\sin\theta\frac{\cos^2\theta - \sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= \frac{r^2}{2}\sin 2\theta\cos 2\theta$$

$$= \frac{r^2}{4}\sin 4\theta$$

$$\therefore \left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| = \left|\frac{r^2}{4}\sin 4\theta\right|$$

$$\therefore \left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| \leqslant \left|\frac{r^2}{4}\right|$$

$$\therefore \left|xy\frac{x^2 - y^2}{x^2 + y^2}\right| \leqslant \frac{x^2 + y^2}{4} \qquad \text{- - - (1)}$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \sqrt{2\epsilon}$ then

$$|x| < \delta,\ |y| < \delta \Rightarrow |x| < \sqrt{2\epsilon},\ |y| < \sqrt{2\epsilon}$$

$$\Rightarrow x^2 + y^2 < 4\epsilon$$

$$\Rightarrow \frac{x^2 + y^2}{4} < \epsilon$$

$$\Rightarrow \left| xy\frac{x^2 - y^2}{x^2 + y^2} \right| < \epsilon \quad \text{(Follows from (1))}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left| xy\frac{x^2 - y^2}{x^2 + y^2} - 0 \right| < \epsilon, \quad \text{whenever} \quad |x - 0| < \delta, |y - 0| < \delta$$

Hence,

$$\lim_{(x,y)\to(0,0)} xy\frac{x^2 - y^2}{x^2 + y^2} = 0$$

13. **Show that** $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = 0$.

Proof:

For any point (x, y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$\frac{xy^2}{x^2 + y^2} = \frac{(r\cos\theta)(r^2\sin^2\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= r\frac{\cos\theta\sin^2\theta}{\cos^2\theta + \sin^2\theta}$$

$$= r\cos\theta\sin^2\theta$$

Therefore, we have,

$$\begin{aligned}\left|\frac{xy^2}{x^2+y^2}\right| &= |r\cos\theta\sin^2\theta| \\ &\leqslant |r||\cos\theta||\sin^2\theta| \\ &\leqslant |r| \\ &= \sqrt{x^2+y^2} \\ &< \sqrt{|x|^2+2|x||y|+|y|^2} \\ &= \sqrt{(|x|+|y|)^2}\end{aligned}$$

$$\therefore \left|\frac{xy^2}{x^2+y^2}\right| < |x|+|y| \quad \text{- - - (1)}$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{2}$ then

$$\begin{aligned}|x|<\delta,\ |y|<\delta &\Rightarrow |x|<\frac{\epsilon}{2},\ |y|<\frac{\epsilon}{2} \\ &\Rightarrow |x|+|y|<\frac{\epsilon}{2}+\frac{\epsilon}{2} \\ &\Rightarrow \left|\frac{xy^2}{x^2+y^2}\right|<\epsilon \quad \text{(Follows from (1))}\end{aligned}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left|\frac{xy^2}{x^2+y^2}-0\right|<\epsilon, \quad \text{whenever} \quad |x-0|<\delta, |y-0|<\delta$$

Hence,

$$\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2}=0$$

14. **Show that** $\displaystyle\lim_{(x,y)\to(0,0)}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=0$

Proof:

Since $(x, y) \to (0,0)$ we can assume x and y both sufficficiently small after some stage.

Now,

$$\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = \frac{[1+x^2y^2]^{\frac{1}{2}}-1}{x^2+y^2}$$

$$= \frac{\left(1+\frac{1}{2}x^2y^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^4y^4+\dots\right)-1}{x^2+y^2}$$

$$= \frac{\frac{1}{2}x^2y^2+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^4y^4+\dots}{x^2+y^2}$$

For sufficiently small values of x and y we can neglect terms starting from second term and get approximation.

$$\therefore \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} \approx \frac{\frac{1}{2}x^2y^2}{x^2+y^2} \text{ - - - } (1)$$

Now, for any point (x, y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$\frac{x^2y^2}{2(x^2+y^2)} = \frac{1}{2}\frac{(r^2\cos^2\theta)(r^2\sin^2\theta)}{r^2\cos^2\theta+r^2\sin^2\theta}$$

$$= \frac{r^2}{2}\left(\frac{\cos^2\theta\sin^2\theta}{\cos^2\theta+\sin^2\theta}\right)$$

$$= \frac{r^2}{8}(4\sin^2\theta\cos^2\theta)$$

$$= \frac{r^2}{8}\sin^2 2\theta$$

$$\leqslant \frac{r^2}{8}$$

$$< r^2$$

$$= x^2+y^2$$

Therefore, we have,

$$\frac{x^2y^2}{2(x^2+y^2)} < x^2+y^2 \text{ - - - } (2)$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \sqrt{\frac{\epsilon}{2}}$ then

$$|x| < \delta,\ |y| < \delta \Rightarrow |x| < \sqrt{\frac{\epsilon}{2}},\ |y| < \sqrt{\frac{\epsilon}{2}}$$

$$\Rightarrow x^2+y^2 < \frac{\epsilon}{2}+\frac{\epsilon}{2}$$

$$\Rightarrow x^2+y^2 < \epsilon$$

$$\Rightarrow \frac{x^2y^2}{2(x^2+y^2)} < \epsilon \quad \text{(Follows from (2))}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left|\frac{x^2y^2}{x^2+y^2} - 0\right| < \epsilon, \quad \text{whenever} \quad |x-0| < \delta, |y-0| < \delta$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^2+y^2} = 0$$

From (1) it follows that,

$$\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$$

15. Show that $\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2}$ exists.

Proof:
For any point (x,y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$\frac{x^3y^3}{x^2+y^2} = \frac{(r^3\cos^3\theta)(r^3\sin^3\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$= r^4\frac{\cos^3\theta\sin^3\theta}{\cos^2\theta+\sin^2\theta}$$

$$= \frac{r^4}{8}(8\cos^3\theta\sin^3\theta)$$

$$= \frac{r^4}{8}(\sin^3 2\theta)$$

$$= \frac{r^4}{8}$$

$$< \frac{r^4}{4}$$

$$= \left(\frac{x^2+y^2}{2}\right)^2$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \sqrt[4]{\epsilon}$ then

$$|x| < \delta,\ |y| < \delta \Rightarrow |x| < \sqrt[4]{\epsilon},\ |y| < \sqrt[4]{\epsilon}$$

$$\Rightarrow x^2 + y^2 < 2\sqrt{\epsilon}$$

$$\Rightarrow (x^2+y^2)^2 < 4\epsilon$$

$$\Rightarrow \frac{(x^2+y^2)^2}{4} < \epsilon$$

$$\Rightarrow \left|\frac{x^3y^3}{x^2+y^2}\right| < \epsilon \quad \text{(Follows from (1))}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left|\frac{x^3y^3}{x^2+y^2} - 0\right| < \epsilon, \quad \text{whenever} \quad |x-0| < \delta, |y-0| < \delta$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^2+y^2} = 0$$

16. **If f and g are two functions defined on some neighbourhood of a point (a, b) such that**

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \textbf{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = m$$

then prove that-

[1]

$$\lim_{(x,y)\to(a,b)} (f+g)(x,y) = l + m$$

Proof

Here,

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Therefore, for each $\epsilon > 0$ there exists some $\delta_1 > 0$ such that

$$|f(x,y) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$$

Also as $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$, for the same ϵ there exists some $\delta_2 > 0$ such that

$$|g(x,y) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x-a| < \delta_2,\ 0 < |y-b| < \delta_2$$

If we take, $\delta = min\{\delta_1, \delta_2\}$ then $\delta \leqslant \delta_1$ and $\delta \leqslant \delta_2$.

Hence,

$$|f(x,y) - l| < \frac{\epsilon}{2} \text{ whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

and

$$|g(x,y) - m| < \frac{\epsilon}{2} \text{ whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

Therefore for $0 < |x-a| < \delta,\ 0 < |y-b| < \delta$,

$$\begin{aligned} |(f(x,y) + g(x,y)) - (l+m)| &= |(f(x,y) - l) + (g(x,y) - m)| \\ &\leqslant |f(x,y) - l| + |g(x,y) - m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$
$$\therefore |(f(x,y) + g(x,y)) - (l+m)| < \epsilon$$

Since

$$|(f(x,y) + g(x,y)) - (l+m)| < \epsilon \text{ whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

we conclude that,

$$\lim_{(x,y)\to(a,b)} [f(x,y) + g(x,y)] = l + m$$

[2]

$$\lim_{(x,y)\to(a,b)}(f-g)(x,y)=l-m$$

Proof:

Here,

$$\lim_{(x,y)\to(a,b)} f(x,y)=l$$

Therefore, for each $\epsilon>0$ there exists some $\delta_1>0$ such that

$$|f(x,y)-l|<\frac{\epsilon}{2}\ \text{ whenever } 0<|x-a|<\delta_1,\ 0<|y-b|<\delta_1$$

Also as $\lim\limits_{(x,y)\to(a,b)} g(x,y)=m$, for the same ϵ there exists some $\delta_2>0$ such that

$$|g(x,y)-m|<\frac{\epsilon}{2}\ \text{ whenever } 0<|x-a|<\delta_2,\ 0<|y-b|<\delta_2$$

If we take, $\delta=min\{\delta_1,\delta_2\}$ then $\delta\leqslant\delta_1$ and $\delta\leqslant\delta_2$.

Hence,

$$|f(x,y)-l|<\frac{\epsilon}{2}\ \text{ whenever } 0<|x-a|<\delta,\ 0<|y-b|<\delta$$

and

$$|g(x,y)-m|<\frac{\epsilon}{2}\ \text{ whenever } 0<|x-a|<\delta,\ 0<|y-b|<\delta$$

Therefore for $0<|x-a|<\delta,\ 0<|y-b|<\delta$

$$\begin{aligned}|(f(x,y)-g(x,y))-(l-m)| &= |(f(x,y)-l)+(m-g(x,y))|\\ &\leqslant |f(x,y)-l|+|m-g(x,y)|\\ &\leqslant |f(x,y)-l|+|g(x,y)-m|\\ &< \frac{\epsilon}{2}+\frac{\epsilon}{2}\\ \therefore |(f(x,y)-g(x,y))-(l-m)| &< \epsilon\end{aligned}$$

Since

$$|(f(x,y)-g(x,y))-(l-m)|<\epsilon \text{ whenever } 0<|x-a|<\delta,\ 0<|y-b|<\delta$$

we conclude that,

$$\lim_{(x,y)\to(a,b)}[f(x,y)-g(x,y)]=l-m$$

[3]

$$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y)=lm$$

Proof

We have,

$$|f(x,y)g(x,y) - lm| = |f(x,y)g(x,y) - g(x,y)l + g(x,y)l - lm|$$

$$= \left|g(x,y)\big(f(x,y) - l\big) + l\big(g(x,y) - m\big)\right|$$

$$\leqslant \left|g(x,y)\big(f(x,y) - l\big)\right| + \left|l\big(g(x,y) - m\big)\right|$$

$$\leqslant |g(x,y)|.|f(x,y) - l| + |l|.|g(x,y) - m|$$

$$\therefore |f(x,y)g(x,y) - lm| \leqslant |g(x,y)|.|f(x,y) - l| + |l|.|g(x,y) - m| \quad \text{- - - (1)}$$

As $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$, for $\epsilon = 1$ there exists some $\delta_1 > 0$ such that

$$|g(x,y) - m| < 1 \quad \text{whenever } 0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$$

Now,

$$\begin{aligned} |g(x,y)| &= |g(x,y) - m + m| \\ &\leqslant |g(x,y) - m| + |m| \\ &< 1 + |m| \quad \text{when } 0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1 \end{aligned}$$

Therefore, $|g(x,y)| \leqslant |m| + 1$ whenever $0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$

So for $0 < |x-a| < \delta_1$ and $0 < |y-b| < \delta_1$, from (1) we have,

$$|f(x,y)g(x,y) - lm| \leqslant (|m|+1).|f(x,y) - l| + |l|.|g(x,y) - m| \quad \text{- - - (2)}$$

Again considering the limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = m$$

for each $\epsilon > 0$ there exists some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x,y) - l| < \frac{\epsilon}{2(|m|+1)} \quad \text{whenever } 0 < |x-a| < \delta_2,\ 0 < |y-b| < \delta_2$$

and

$$|g(x,y) - m| < \frac{\epsilon}{2(|l|+1)} \quad \text{whenever } 0 < |x-a| < \delta_3,\ 0 < |y-b| < \delta_3$$

If we take, $\delta = min\{\delta_1, \delta_2, \delta_3\}$ then $\delta \leqslant \delta_1$, $\delta \leqslant \delta_2$ and $\delta \leqslant \delta_3$.
Hence,

$$|f(x,y) - l| < \frac{\epsilon}{2(|m|+1)} \quad \text{whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

and

$$|g(x,y) - m| < \frac{\epsilon}{2(|l|+1)} \quad \text{whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

Therefore, for $0 < |x-a| < \delta,\ 0 < |y-b| < \delta$ from (2) it follows that,

$$\begin{aligned} |f(x,y)g(x,y) - lm| &\leqslant (|m|+1).\frac{\epsilon}{2(|m|+1)} + |l|.\frac{\epsilon}{2(|l|+1)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \end{aligned}$$

$$\therefore |f(x,y)g(x,y) - lm| < \epsilon$$

Since

$$|f(x,y)g(x,y) - lm| < \epsilon \text{ whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

we conclude that,

$$\lim_{(x,y)\to(a,b)} f(x,y)g(x,y) = lm$$

[4]

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{l}{m}, \textbf{ provided } m \neq 0, \quad \textbf{when} \quad (x,y)\to(a,b)$$

Proof

We have,

$$\left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| = \left|\frac{mf(x,y) - lg(x,y)}{mg(x,y)}\right|$$

$$= \left|\frac{mf(x,y) - lm + lm - lg(x,y)}{mg(x,y)}\right|$$

$$= \frac{\left|m\big(f(x,y) - l\big) + l\big(m - g(x,y)\big)\right|}{|m||g(x,y)|}$$

$$\leqslant \frac{|m||f(x,y) - l|}{|m||g(x,y)|} + \frac{|l||g(x,y) - m|}{|m||g(x,y)|}$$

$$\therefore \left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| \leqslant \frac{1}{|g(x,y)|}.|f(x,y) - l| + \frac{|l|}{|m||g(x,y)|}|g(x,y) - m| \text{ - - - (1)}$$

As $m \neq 0$ we have $|m| > 0$, hence $\frac{|m|}{2} > 0$

Since, $\lim\limits_{(x,y)\to(a,b)} g(x,y) = m$ there exists some $\delta_1 > 0$ such that

$$|g(x,y) - m| < \frac{|m|}{2} \text{ whenever } 0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$$

Now,

$$\begin{aligned} |m| &= |m - g(x,y) + g(x,y)| \\ &\leqslant |g(x,y) - m| + |g(x,y)| \\ &\leqslant \frac{|m|}{2} + |g(x,y)| \\ |m| - \frac{|m|}{2} &< |g(x,y)| \\ \therefore \frac{|m|}{2} &\leqslant |g(x,y)| \end{aligned}$$

Therefore, $\dfrac{1}{|g(x,y)|} \leqslant \dfrac{2}{|m|}$ whenever $0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$

So for $0 < |x-a| < \delta_1,\ 0 < |y-b| < \delta_1$, from (1) we have,

$$\left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| \leqslant \frac{2}{|m|}|f(x,y) - l| + \frac{2|l|}{|m|^2}|g(x,y) - m| \quad \text{- - - (2)}$$

Again we consider the limits

$$\lim_{(x,y)\to(a,b)} f(x,y) = l \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = m$$

Therefore, for each $\epsilon > 0$ there exists some $\delta_2 > 0$ and $\delta_3 > 0$ such that

$$|f(x,y) - l| < \frac{\epsilon|m|}{4} \quad \text{whenever } 0 < |x-a| < \delta_2,\ 0 < |y-b| < \delta_2$$

and

$$|g(x,y) - m| < \frac{\epsilon|m|^2}{4(|l|+1)} \quad \text{whenever } 0 < |x-a| < \delta_3,\ 0 < |y-b| < \delta_3$$

If we take, $\delta = min\{\delta_1, \delta_2, \delta_3\}$ then $\delta \leqslant \delta_1$, $\delta \leqslant \delta_2$ and $\delta \leqslant \delta_3$.
Hence,

$$|f(x,y) - l| < \frac{\epsilon|m|}{4} \quad \text{whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

and

$$|g(x,y) - m| < \frac{\epsilon|m|^2}{4(|l|+1)} \quad \text{whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

Therefore for $0 < |x-a| < \delta,\ 0 < |y-b| < \delta$ from (2) it follows that,

$$\begin{aligned}\left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| &< \frac{2}{|m|}\left(\frac{\epsilon|m|}{4}\right) + \frac{2|l|}{|m|^2}\left(\frac{\epsilon|m|^2}{4(|l|+1)}\right)\\ &< \frac{\epsilon}{2} + \left(\frac{|l|}{|l|+1}\right)\frac{\epsilon}{2}\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2}\\ \therefore \left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| &< \epsilon\end{aligned}$$

Since

$$\left|\frac{f(x,y)}{g(x,y)} - \frac{l}{m}\right| < \epsilon \text{ whenever } 0 < |x-a| < \delta,\ 0 < |y-b| < \delta$$

we conclude that,

$$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{l}{m}$$

17. Prove that $\lim\limits_{(x,y)\to(1,2)} (x^2 + 2y) = 5$.

Proof

To prove the result using the definition of limit of a function, we shall show that for any given $\epsilon > 0$ there exists some $\delta > 0$ such that,

$$|x^2 + 2y - 5| < \epsilon \text{ whenever } 0 < |x-1| < \delta,\ 0 < |y-2| < \delta$$

Now, for any $\delta > 0$, if $|x-1| < \delta$ and $|y-2| < \delta$ then $x \neq 1,\ y \neq 2$ and

$$1 - \delta < x < 1 + \delta \text{ and } 2 - \delta < y < 2 + \delta$$

$$\therefore \quad 1 - 2\delta + \delta^2 < x^2 < 1 + 2\delta + \delta^2 \text{ and } 4 - 2\delta < 2y < 4 + 2\delta$$

$$\therefore \quad 5 - 4\delta + \delta^2 < x^2 + 2y < 5 + 4\delta + \delta^2$$

$$\therefore \quad -4\delta + \delta^2 < x^2 + 2y - 5 < 4\delta + \delta^2$$

Taking $0 < \delta \leqslant 1$ we have $\delta^2 \leqslant \delta$

Hence,

$$-4\delta - \delta < x^2 + 2y - 5 < 4\delta + \delta$$

$$\therefore \quad -5\delta < x^2 + 2y - 5 < 5\delta$$

$$\therefore \quad |(x^2 + 2y) - 5| < 5\delta$$

So, for any given $\epsilon > 0$ if we take $\delta = min\left\{1, \frac{\epsilon}{5}\right\}$ then,

$$|x^2 + 2y - 5| < \epsilon \text{ whenever } 0 < |x-1| < \delta,\ 0 < |y-2| < \delta$$

Hence,

$$\lim_{(x,y)\to(1,2)} (x^2 + 2y) = 5$$

18. Prove that $\lim\limits_{(x,y)\to(1,2)} 3xy = 6$.

Proof

To prove the result using the definition of limit of a function, we shall show that for any given $\epsilon > 0$ there exists some $\delta > 0$ such that,

$$|3xy - 6| < \epsilon \text{ whenever } 0 < |x-1| < \delta,\ 0 < |y-2| < \delta$$

Now, for any $0 < \delta \leqslant 1$, if $0 < |x-1| < \delta$ and $0 < |y-2| < \delta$ then $x \neq 1,\ y \neq 2$ and

$$
\begin{aligned}
& & 1-\delta < x < 1+\delta \ \text{ and } \ 2-\delta < y < 2+\delta \\
\therefore & & 3(1-\delta)(2-\delta) < 3xy < 3(1+\delta)(2+\delta) \\
\therefore & & 3(2-3\delta+\delta^2) < 3xy < 3(2+3\delta+\delta^2) \\
\therefore & & 6-9\delta+3\delta^2 < 3xy < 6+9\delta+3\delta^2 \\
\therefore & & -9\delta+3\delta^2 < 3xy-6 < 9\delta+3\delta^2 \\
\therefore & & -9\delta-3\delta < 3xy-6 < 9\delta+3\delta \\
\therefore & & -12\delta < 3xy-6 < 12\delta \\
\therefore & & |3xy-6| < 12\delta
\end{aligned}
$$

So, for any given $\epsilon > 0$ if we take $\delta = min\left\{1, \frac{\epsilon}{12}\right\}$ then,

$$|3xy-6| < \epsilon \ \text{ whenever } \ 0 < |x-1| < \delta,\ 0 < |y-2| < \delta$$

Hence,

$$\lim_{(x,y)\to(1,2)} 3xy = 6$$

19. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2} = 0.$

Answer:

If $(x,y)\to(0,0)$ then $x^2+y^2\to 0$. So,

$$\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2} = \lim_{(x,y)\to(0,0)} x \lim_{(x,y)\to(0,0)} \cdot \frac{\sin(x^2+y^2)}{x^2+y^2} = 0.1 = 0$$

20. Show that $\displaystyle\lim_{((x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{tan^{-1}(3xy-6)} = \frac{1}{3}.$

Proof

Let $xy-2=t$. Now, if $(x,y)\to(2,1)$ then $(xy-2)\to 0$, hence $t\to 0$.

Therefore,

$$\lim_{(x,y)\to(2,1)} \frac{\sin^{-1}(xy-2)}{(3xy-6)} = \lim_{t\to 0} \frac{\sin^{-1} t}{tan^{-1}3t} = \lim_{t\to 0} \frac{\frac{1}{\sqrt{1-t^2}}}{3\frac{1}{1+9t^2}} = \frac{1}{3}$$

21. Show that $\displaystyle\lim_{((x,y)\to(0,1)} e^{\frac{-1}{x^2((y-1)^2)}} = 0.$

Proof

$$
\begin{aligned}
(x,y)\to(0,1) &\Rightarrow x\to 0,\ y\to 1\\
&\Rightarrow x^2\to 0+,\ (y-1)^2\to 0+\\
&\Rightarrow x^2(y-1)^2\to 0+\\
&\Rightarrow \frac{1}{x^2(y-1)^2}\to\infty
\end{aligned}
$$

So, if we take $\dfrac{1}{x^2(y-1)^2}=t$ then $t\to\infty$

As, $e>1$, we have, $t\to\infty\Rightarrow e^t\to\infty$

Therefore,

$$\lim_{(x,y)\to(0,1)} e^{-1/x^2(y-1)^2}=\lim_{t\to\infty}e^{-t}=\lim_{t\to\infty}\frac{1}{e^t}=0$$

22. **Show that for the function $f(x,y)=\dfrac{xy}{x^2+y^2}$ the repeated limits exist and are equal at the origin but the simultaneous limit does not exist.**

Proof

For $f(x,y)=\dfrac{xy}{x^2+y^2}$ we have,

$$\lim_{y\to0}\lim_{x\to0}f(x,y)=\lim_{y\to0}\lim_{x\to0}\frac{xy}{x^2+y^2}=\lim_{y\to0}\frac{0}{y^2}=0$$

and

$$\lim_{x\to0}\lim_{y\to0}f(x,y)=\lim_{x\to0}\lim_{y\to0}\frac{xy}{x^2+y^2}=\lim_{x\to0}\frac{0}{x^2}=0$$

Thus, both the repeated limits exist.

Next consider any non-zero constant m and let us evaluate the limit through the path $y=mx$

$$
\begin{aligned}
\lim_{x\to0}f(x,mx)&=\lim_{x\to0}\frac{x(mx)}{x^2+m^2x^2}\\
&=\lim_{x\to0}\frac{mx^2}{x^2(1+m^2)}\\
&=\lim_{x\to0}\frac{m}{1+m^2}\\
\therefore \lim_{(x,y)\to(0,0)}f(x,mx)&=\frac{m}{1+m^2}
\end{aligned}
$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1 x$ and $y = m_2 x$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, m_1 x) = \frac{m_1}{1+m_1^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, m_2 x) = \frac{m_2}{1+m_2^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, m_1 x) \neq \lim_{(x,y)\to(0,0)} f(x, m_2 x)$$

Hence, the simultaneous limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

23. **Show that for the function** $f(x,y) = \begin{cases} x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}, & \textbf{when} \quad xy \neq 0 \\ 0, & \textbf{when} \quad xy = 0 \end{cases}$ **the simultaneous limit exists at the origin but the repeated limits do not exist.**

Proof

If $x \to 0-$ then $\frac{1}{x} \to -\infty$ and if $x \to 0+$ then $\frac{1}{x} \to \infty$

Therefore $\lim\limits_{x\to 0}\left(x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}\right)$ does not exist.

Therefore, $\lim\limits_{y\to 0}\lim\limits_{x\to 0}\left(x\sin\dfrac{1}{y} + y\sin\dfrac{1}{x}\right)$ does not exist.

Hence, $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$ does not exist.

Similarly $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$ does not exist.

Now,

$$\left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| \leqslant |x|\left|\sin\frac{1}{y}\right| + |y|\left|\sin\frac{1}{x}\right| \leqslant |x| + |y|$$

Therefore, for any $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{2}$ then

$$\begin{aligned} |x-0| < \delta,\ |y-0| < \delta &\Rightarrow |x| < \frac{\epsilon}{2},\ |y| < \frac{\epsilon}{2} \\ &\Rightarrow |x| + |y| < \epsilon \\ &\Rightarrow \left|x\sin\frac{1}{y} + y\sin\frac{1}{x}\right| < \epsilon \end{aligned}$$

Hence,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Thus, both the repeated limits do not exist but the simultaneous limit exists.

24. **Show that for the function $f(x,y) = \begin{cases} 1, & \text{if } xy \neq 0 \\ 0, & \text{if } xy = 0 \end{cases}$ the repeated limits exist at the origin and are equal but the the simultaneous limit does not exist.**

Answer:

We know that $x \neq 0$ when $x \to 0$. Therefore dependeing on whether $y = 0$ or $y \neq 0$ we have,

$$\lim_{x \to 0} f(x,y) = \begin{cases} 1, \ if y \neq 0 & (\because xy \neq 0 \text{ as } x \neq 0, \ y \neq 0) \\ 0, \ if y = 0 & (\because xy = 0 \text{ as } x \neq 0, but \ y = 0) \end{cases}$$

Since $y \neq 0$ when $y \to 0$, we get,

$$\lim_{y \to 0} \lim_{x \to 0} f(x,y) = 1$$

Simlarly,

$$\lim_{x \to 0} \lim_{y \to 0} f(x,y) = 1$$

Hence, the repeated limits exist and are equal.

Now, for the simultaneous limit consider any neighnourhood of $(0,0)$. For any point on the X-axis we have $y = 0$ and for any point on the Y-axis we have $x = 0$. Therefore, for all the points on any axis we have $xy = 0$ hence $f(x,y) = 0$ at these points. Also at all other points we have $xy \neq 0$ hence $f(x,y) = 1$ at rest of the points.

Thus there is some $0 < \epsilon$ such that

$$|f(x,y) - f(0,0)| = |f(x,y)| \not< \epsilon$$

for all points in the neighbourhood.

Hence, $\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does not exist.

25. **Continuity of a function.**

Continuity of a function

A function f is said to be continuous at a point (a,b) of its domain if

$$\lim_{(x,y) \to (a,b)} f(x,y) = f(a,b)$$

26. **Investigate the continuity at $(0,0)$ of $f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.**

For any non-zero constant m let us evaluate the limit through the path $y = mx$

$$\lim_{x\to 0} f(x, mx) = \lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$
$$= \lim_{x\to 0} \frac{1 - m^2}{1 + m^2}$$
$$\therefore \lim_{(x,y)\to(0,0)} f(x, mx) = \frac{1 - m^2}{1 + m^2}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x$ and $y = m_2x$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) = \frac{1 - m_1^2}{1 + m_1^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, m_2x) = \frac{1 - m_2^2}{1 + m_2^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, m_1x) \neq \lim_{(x,y)\to(0,0)} f(x, m_2x)$$

Hence, the simultaneous limit $\lim_{(x,y)\to(0,0)} f(x, y)$ does not exist. Therefore, f is not continuous at $(0, 0)$.

27. **Show that the function** $f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ **is continuous at the origin.**

Proof:
For any point (x, y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$\begin{aligned}\frac{xy}{\sqrt{x^2+y^2}} &= \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}\\ &= r\frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+\sin^2\theta}}\\ &= \frac{r}{2}(2\cos\theta\sin\theta)\\ &= \frac{r}{2}(\sin 2\theta)\\ &\leqslant r\\ \therefore\ \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leqslant r\end{aligned}$$

$$\begin{aligned} &= \sqrt{x^2+y^2}\\ &< \sqrt{|x|^2+2|x||y|+|y|^2}\\ &= \sqrt{(|x|+|y|)^2}\\ \therefore\ \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &< |x|+|y| \quad \text{- - - (1)}\end{aligned}$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{2}$ then

$$\begin{aligned}|x|<\delta,\ |y|<\delta &\Rightarrow |x|<\frac{\epsilon}{2},\ |y|<\frac{\epsilon}{2}\\ &\Rightarrow |x|+|y|<\frac{\epsilon}{2}+\frac{\epsilon}{2}\\ &\Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon \quad \text{(Follows from (1))}\end{aligned}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon, \quad \text{whenever} \quad |x-0|<\delta, |y-0|<\delta$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence, f is continuous at the origin.

28. Partial Derivatives

Partial Derivatives:

For a function $f(x,y)$ if $\lim\limits_{\delta x\to 0}\dfrac{f(x+\delta x,y)-f(x,y)}{\delta x}$ exists then it is known as the partial derivative of f with respect to x and it is generally denoted by $\dfrac{\partial f}{\partial x}$ or $f_x(x,y)$. Thus,

$$\frac{\partial f}{\partial x}=\lim_{\delta x\to 0}\frac{f(x+\delta x,y)-f(x,y)}{\delta x}$$

Similary the partial derivative with respect to y, if exists, is given by

$$\frac{\partial f}{\partial y}=\lim_{\delta y\to 0}\frac{f(x,y+\delta y)-f(x,y)}{\delta y}$$

29. If $f(x,y)=2x^2-xy+2y^2$ then find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at $(1,2)$.

Answer:
For $f(x,y)=2x^2-xy+2y^2$ we have, $f_x(x,y)=4x-y$ and $f_y(x,y)=-x+4y$

Therefore, $f_x(1,2)=4(1)-2=2$ and $f_y(1,2)=-1+8=7$

30. Show that the function $f(x,y)=\begin{cases}\dfrac{x^2y}{x^4+y^2}, & x^2+y^2\neq 0\\ 0, & x=0=y\end{cases}$

possesses first partial derivatives everywhere, including the origin, but the function is discontinuous at the origin.

Answer:
For $(x,y)\neq(0,0)$

$$\frac{\partial}{\partial x}\left(\frac{x^2y}{x^4+y^2}\right)=y\left(\frac{(x^4+y^2)(2x)-x^2(4x^3)}{(x^4+y^2)^2}\right)=\frac{y(2xy^2-2x^5)}{(x^4+y^2)^2}$$

Also,

$$\frac{\partial}{\partial y}\left(\frac{x^2y}{x^4+y^2}\right)=x^2\left(\frac{(x^4+y^2)(1)-y(2y)}{(x^4+y^2)^2}\right)=\frac{x^2(x^4-y^2)}{(x^4+y^2)^2}$$

Thus $f(x,y)$ possesses both first partial derivatives $f_x(x,y)$ and $f_y(x,y)$ at $(x,y)\neq(0,0)$

Now,

$$\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{h(0)-0}{h}=0$$

and

$$\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{k(0)-0}{h}=0$$

Hence, $f_x(0,0) = 0$ and $f_y(0,0) = 0$.

Thus, $f(x,y)$ possesses first partial derivatives everywhere, including the origin.

Next let us show that $f(x,y)$ is not continuous at the origin. For a non-zero constant m let us evaluate the limit through the path $y = mx^2$

$$\begin{aligned}\lim_{x\to 0} f(x, mx^2) &= \lim_{x\to 0} \frac{x^2(mx^2)}{x^4 + m^2x^4} \\ &= \lim_{x\to 0} \frac{mx^4}{x^4(1+m^2)} \\ &= \lim_{x\to 0} \frac{m}{1+m^2} \\ \therefore \lim_{(x,y)\to(0,0)} f(x, mx^2) &= \frac{m}{1+m^2}\end{aligned}$$

This implies that for two distinct non-zero constants m_1 and m_2 if we evaluate the limits through $y = m_1x^2$ and $y = m_2x^2$ then we get

$$\lim_{(x,y)\to(0,0)} f(x, m_1x^2) = \frac{m_1}{1+m_1^2} \quad \text{and} \quad \lim_{(x,y)\to(0,0)} f(x, m_2x^2) = \frac{m_2}{1+m_2^2}$$

But then

$$\lim_{(x,y)\to(0,0)} f(x, m_1x^2) \neq \lim_{(x,y)\to(0,0)} f(x, m_2x^2)$$

Therefore, the simultaneous limit $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Hence, $f(x,y)$ is not continuous at the origin.

31. **If $f(x,y) = \sqrt{|xy|}$, find $f_x(0,0)$ and $f_y(0,0)$.**

Answer:

$$\lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{|h(0)| - 0}{h} = 0$$

Therefore, $f_x(0,0) = 0$

Also,

$$\lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k\to 0} \frac{|k(0)| - 0}{k} = 0$$

Therefore, $f_y(0,0) = 0$

32. **If f_x exists throughout a neighbourhood of a point (a,b) and $f_y(a,b)$ exists then for any point $(a+h, b+k)$ of this neighbourhood**

$$f(a+h, b+k) - f(a,b) = hf_x(a+\theta h, b+k) + k[f_y(a,b) + \eta]$$

where $0 < \theta < 1$ and η is a function of k, tending to zero with k.

Proof

We can write,

$$f(a+h,b+k)-f(a,b)=f(a+h,b+k)-f(a,b+k)+f(a,b+k)-f(a,b) \text{ - - - (1)}$$

Now, f_x exists throughout a neighbourhood of (a,b). Therefore, by Lagrange's Mean Value theorem, for some $0<\theta<1$ we have,

$$f(a+h,b+k)-f(a,b+k)=hf_x(a+\theta h,b+k) \text{ - - - (2)}$$

Moreover, $f_y(a,b)$ exists. Therefore,

$$\lim_{k\to 0}\frac{f(a,b+k)-f(a,b)}{k}=f_y(a,b)$$

Therefore,

$$f(a,b+k)-f(a,b)=k[f_y(a,b)+\eta] \text{ - - - (3)}$$

where η is a function of k such that it $\lim\limits_{k\to 0}\eta=0$.

Substituting from (2) and (3) in (1) we get,

$$f(a+h,b+k)-f(a,b)=hf_x(a+\theta h,b+k)+k[f_y(a,b)+\eta])$$

where $0<\theta<1$ and η is a function of k, tending to zero with k.

33. State and prove a sufficient condition for a function $f(x,y)$ to be continuous at a point (a,b).

Sufficient condition for a function $f(x,y)$ to be continuous at a point (a,b)

A sufficient condition that a function f is continuous at a point (a,b) is that one of its partial derivatives exists and is bounded in a neighbourhood of (a,b) and that the other exists at (a,b)

Proof

Suppose $f_x(x,y)$ exists and it is bounded in some neighbourhood of (a,b). Also suppose $f_y(a,b)$ exist then for any point $(a+h,b+k)$ of this neighbourhood by above Mean Value therorem,

$$f(a+h,b+k)-f(a,b)=hf_x(a+\theta h,b+k)+k[f_y(a,b)+\eta]$$

where $0<\theta<1$ and η is a function of k, tending to zero with k. We note that $f_x(a+\theta h,b+k)$ is bounded.

Now,

$$\begin{aligned}
\lim_{(h,k)\to(0,0)} f(a+h,b+k)-f(a,b) &= \lim_{(h,k)\to(0,0)}[f(a+h,b+k)-f(a,b)]\\
&= \lim_{(h,k)\to(0,0)}[hf_x(a+\theta h,b+k)+k(f_y(a,b)+\eta)]\\
&= \lim_{(h,k)\to(0,0)} hf_x(a+\theta h,b+k)+\lim_{(h,k)\to(0,0)} k[f_y(a,b)+\eta]\\
&= \lim_{(h,k)\to(0,0)} hf_x(a+\theta h,b+k)+\lim_{(h,k)\to(0,0)}[kf_y(a,b)+k\eta]\\
&= 0+0\\
\therefore \lim_{(h,k)\to(0,0)} f(a+h,b+k) &= f(a,b)
\end{aligned}$$

Hence, f is continuous at (a, b).

34. Differentiability and Differential.

Differentiability:

Let (x, y) and $(x+\delta x, y+\delta y)$ be two neighnouring points in the domain of a function $f(x, y)$. The change δf in the function as the point changes from (x, y) to $(x+\delta x, y+\delta y)$ is given by,

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y)$$

The finction f is said to be differentiable at (x, y) if the change δf can be expressed in the form

$$\delta f = A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y)$$

where A and B are constants independent of δx and δy and ϕ, ψ are functions of δx and δy both of which tend to zero as δx and δy tend to zero simultaneously.

Here, $A\delta x + B\delta y$ is called the differential of f at (x, y) and it is denoted by

$$df = A\delta x + B\delta y$$

35. Prove that if a function $f(x, y)$ is differentiable then it is continuous and both the first order partial derivatives exist.

Proof

Suppose a function $f(x, y)$ be differentiable at a point (x, y). Therefore if (x, y) and $(x+\delta x, y+\delta y)$ are two neighbouring points in the domain of f then the change δf in f as (x, y) changes from (x, y) to $(x+\delta x, y+\delta y)$ given by

$$\delta f = f(x+\delta x, y+\delta y) - f(x, y)$$

can be expressed in the form

$$\delta f = A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y) \text{ - - - (1)}$$

where A and B are constants independent of δx and δy and

$$\lim_{(\delta x,\delta y)\to(0,0)} \phi(\delta x, \delta y) = 0 \quad \text{and} \quad \lim_{(\delta x,\delta y)\to(0,0)} \phi(\delta x, \delta y) = 0$$

Therefore,

$$\begin{aligned}
&\lim_{(\delta x,\delta y)\to(0,0)} f(x+\delta x, y+\delta y) - f(x, y) \\
&= \lim_{(\delta x,\delta y)\to(0,0)} [f(x+\delta x, y+\delta y) - f(x, y)] \\
&= \lim_{(\delta x,\delta y)\to(0,0)} [A\delta x + B\delta y + \delta x\phi(\delta x, \delta y) + \delta y\psi(\delta x, \delta y)] \quad \text{(From (1))} \\
&= A \lim_{(\delta x,\delta y)\to(0,0)} \delta x + B \lim_{(\delta x,\delta y)\to(0,0)} \delta y + \lim_{(\delta x,\delta y)\to(0,0)} \delta x\phi(\delta x, \delta y) + \lim_{(\delta x,\delta y)\to(0,0)} \delta y\psi(\delta x, \delta y) \\
&= A(0) + B(0) + 0 + 0 \\
&= 0
\end{aligned}$$

Therefore,

$$\lim_{(\delta x,\delta y)\to(0,0)} f(x+\delta x, y+\delta y) = f(x,y)$$

Hence, f is continuous at (x, y).

Also, if we keep y constant then $\delta y = 0$, hence

$$\delta f = A\delta x + \delta x\phi(\delta x, 0)$$

Therefore.

$$\frac{\delta f}{\delta x} = A + \phi(\delta x, 0)$$

Therefore,

$$\lim_{\delta x\to 0} \frac{\delta f}{\delta x} = \lim_{\delta x\to 0} A + 0 = A$$

Hence, $f_x(x, y)$ exists and

$$\frac{\partial f}{\partial x} = A$$

Similarly, if we keep x constant then $\delta x = 0$, hence

$$\delta f = B\delta y + \delta y\psi(0, \delta y)$$

Therefore,

$$\frac{\delta f}{\delta y} = B + \psi(0, \delta y)$$

Therefore,

$$\lim_{\delta y\to 0} \frac{\delta f}{\delta y} = \lim_{\delta y\to 0} B + 0 = B$$

Hence, $f_y(x, y)$ exists and

$$\frac{\partial f}{\partial y} = B$$

36. **Show that the function** $f(x,y) = \begin{cases} \dfrac{x^3 - y^3}{x^2 + y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ **is continuous and possesses partial derivatives of first order at** $(0,0)$ **but is not differentiable at** $(0,0)$**.**

Answer:
First we show that f is continuous at $(0, 0)$.

Take $x = r\cos\theta$ and $y = r\sin\theta$.

Therefore,

$$\begin{aligned}\left|\frac{x^3-y^3}{x^2+y^2}\right| &= \left|\frac{r^3\cos^3\theta - r^3\sin^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right| \\ &= r\left|\frac{\cos^3\theta - \sin^3\theta}{\cos^2\theta+\sin^2\theta}\right| \\ &\leqslant r(\left|\cos^3\theta\right| + \left|\sin^3\theta\right| \\ &\leqslant 2r \\ \therefore \left|\frac{x^3-y^3}{x^2+y^2}\right| &\leqslant 2\sqrt{x^2+y^2} \text{ - - - } (1)\end{aligned}$$

So, for any given $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{2\sqrt{2}}$ then

$$\begin{aligned}|x-0|<\delta,\ |y-0|<\delta &\Rightarrow |x| < \frac{\epsilon}{2\sqrt{2}},\ |y| < \frac{\epsilon}{2\sqrt{2}} \\ &\Rightarrow x^2+y^2 < \frac{\epsilon^2}{4} \\ &\Rightarrow 2\sqrt{x^2+y^2} < \epsilon \\ &\Rightarrow \left|\frac{x^3-y^3}{x^2+y^2}\right| < \epsilon \qquad \text{(From (1))}\end{aligned}$$

Hence,

$$\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$$

As $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$, f is continuous at $(0,0)$.

Next, we show that $f_x(0,0)$ and $f_y(0,0)$ both exist.

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\dfrac{h^3-0}{h^2+0}-0}{h} = 1$$

Also,

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{\dfrac{0-k^3}{0+k^2}-0}{k} = -1$$

Thus, f possesses both the first order partial derivatives and

$$f_x(0,0) = 1 \quad \text{and} \quad f_y(0,0) = -1$$

Finally, we show that f is not differentiable at $(0,0)$.

If possible suppose f is differentiable at $(0,0)$. Therefore, the change δf in f as (x,y) changes from $(0,0)$ to (h,k) is given by

$$\delta f = f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi$$

where, $A = f_x(0,0) = 1$ and $B = f_y(0,0) = -1$ and $\lim\limits_{(h,k)\to(0,0)} \phi = 0$ and $\lim\limits_{(h,k)\to(0,0)} \psi = 0$

Therefore,

$$\frac{h^3-k^3}{h^2+k^2} = h - k + h\phi + k\psi$$

Now, for $x = \rho\cos\theta$ and $y = \rho\sin\theta$, we get,

$$\begin{aligned}
\frac{\rho^3\cos^3\theta - \rho^3\sin^3\theta}{\rho^2\cos^2\theta + \rho^2\sin^2\theta} &= \rho\cos\theta - \rho\sin\theta + \rho\phi\cos\theta + \rho\psi\sin\theta \\
\therefore\ \frac{\cos^3\theta - \sin^3\theta}{\cos^2\theta + \sin^2\theta} &= r\left(\cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta\right) \\
\therefore\ \cos^3\theta - \sin^3\theta &= \cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta \text{ - - - } (1)
\end{aligned}$$

Now, $\dfrac{k}{h} = \dfrac{\rho\sin\theta}{\rho\cos\theta} = tan\theta.$

Therefore, for any $\theta = tan^{-1}\dfrac{k}{h}$, $\rho \to 0$ implies that $(h,k) \to (0,0)$. Therefore, from (1) we get,

$$\begin{aligned}
\lim_{(h,k)\to(0,0)} \cos^3\theta - \sin^3\theta &= \lim_{(h,k)\to(0,0)} (\cos\theta - \sin\theta + \phi\cos\theta + \psi\sin\theta) \\
\cos^3\theta - \sin^3\theta &= \cos\theta - \sin\theta + \cos\theta \lim_{(h,k)\to(0,0)} \phi + \sin\theta \lim_{(h,k)\to(0,0)} \psi \\
&= \cos\theta - \sin\theta \\
\cos\theta(1-\cos^2\theta) - \sin\theta(1-\sin^2\theta) &= 0 \\
\cos\theta\sin\theta(\sin\theta - \cos\theta) &= 0
\end{aligned}$$

which is not possible for all θ. Therefore, f is not differentiable at $(0,0)$

37. **Show that the function** $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x = 0 = y \end{cases}$ **is continuous and possesses partial derivatives of first order at** $(0,0)$ **but is not differentiable at** $(0,0)$**.**

Answer:
First we show that f is continuous at $(0,0)$.

For any point (x,y) we can take some $r > 0$ and a real number θ such that

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

Therefore,

$$\begin{aligned}
\frac{xy}{\sqrt{x^2+y^2}} &= \frac{(r\cos\theta)(r\sin\theta)}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}} \\
&= r\frac{\cos\theta\sin\theta}{\sqrt{\cos^2\theta+\sin^2\theta}} \\
&= \frac{r}{2}(2\cos\theta\sin\theta) \\
&= \frac{r}{2}(\sin 2\theta) \\
&\leqslant r \\
\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &\leqslant r
\end{aligned}$$

$$\begin{aligned}
&= \sqrt{x^2+y^2} \\
&< \sqrt{|x|^2+2|x||y|+|y|^2} \\
&= \sqrt{(|x|+|y|)^2} \\
\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| &< |x|+|y| \quad \text{- - - (1)}
\end{aligned}$$

Therefore, for any given $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{2}$ then

$$\begin{aligned}
|x| < \delta,\ |y| < \delta &\Rightarrow |x| < \frac{\epsilon}{2},\ |y| < \frac{\epsilon}{2} \\
&\Rightarrow |x|+|y| < \frac{\epsilon}{2}+\frac{\epsilon}{2} \\
&\Rightarrow \left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon \quad \text{(Follows from (1))}
\end{aligned}$$

Thus, for any given $\epsilon > 0$ we can find some $\delta > 0$ such that

$$\left|\frac{xy}{\sqrt{x^2+y^2}}\right| < \epsilon, \quad \text{whenever} \quad |x-0| < \delta, |y-0| < \delta$$

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$$

Hence, f is continuous at the origin.

Next, we show that $f_x(0,0)$ and $f_y(0,0)$ both exist.

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\dfrac{h(0)-0}{\sqrt{h^2+0}}-0}{h} = 0$$

Also,

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}=\lim_{k\to 0}\frac{\dfrac{k(0)-0}{\sqrt{0+k^2}}-0}{k}=0$$

Thus, f possesses both the first order partial derivatives and

$$f_x(0,0)=0 \quad \text{and} \quad f_y(0,0)=0$$

Finally, we show that, f is not differentiable at $(0,0)$.

If possible suppose f is differentiable at $(0,0)$. Therefore, the change δf in f as (x,y) changes from $(0,0)$ to (h,k) is given by

$$\delta f = f(h,k)-f(0,0)=Ah+Bk+h\phi+k\psi$$

where, $A=f_x(0,0)=0$ and $B=f_y(0,0)=0$ and $\lim\limits_{(h,k)\to(0,0)}\phi=0$ and $\lim\limits_{(h,k)\to(0,0)}\psi=0$

Therefore,

$$\frac{hk}{\sqrt{h^2+k^2}}=h(0)+k(0)+h\phi+k\psi$$

Therefore,

$$\frac{hk}{\sqrt{h^2+k^2}}=h\phi+k\psi$$

If we take $k=mh$ and let $h\to 0$ then we have $k\to 0$

Now,

$$\begin{aligned}
\lim_{(h,k)\to(0,0)}\frac{hk}{\sqrt{h^2+k^2}} &= \lim_{(h,k)\to(0,0)}(h\phi+k\psi)\\
\therefore \lim_{h\to 0}\frac{h(mh)}{\sqrt{h^2+m^2h^2}} &= \lim_{h\to 0}(h\phi+mh\psi)\\
\therefore \lim_{h\to 0}\frac{mh}{\sqrt{1+m^2}} &= \lim_{h\to 0}h(\phi+m\psi)\\
\therefore \lim_{h\to 0}\frac{m}{\sqrt{1+m^2}} &= \lim_{h\to 0}(\phi+m\psi)\\
\therefore \frac{m}{\sqrt{1+m^2}} &= 0
\end{aligned}$$

which is not true for every m. Therefore, f is not differentiable at $(0,0)$

38. State and prove a sufficient condition for differentiablity of a function.

Sufficient condition for differentiablity of a function:

If (a,b) is a point in the domain of a function f such that
(1) f_x is continuous at (a,b) and
(2) f_y exists at (a,b)
then f is differentiable at (a,b).

Proof:

Continuity of f_x at (a,b) implies that f_x exists in some neighbourhood $(a-\delta, a+\delta; b-\delta, b+\delta)$ of (a,b).

Let $(a+h, b+k)$ be a point in this neighbourhood.

Now, the change δf while point changes from (a,b) to $(a+h, b+k)$ is given by, $\delta f = f(a+h, b+k) - f(a,b)$

$$\therefore \; \delta f = f(a+h, b+k) - f(a, b+k) + f(a, b+k) - f(a,b) \text{ - - - } (1)$$

Since, f_x exists in $(a-\delta, a+\delta;\; b-\delta, b+\delta)$, by the Lagrange's Mean Value theorem we get,

$$f(a+h, b+k) - f(a, b+k) = hf_x(a+\theta h, b+k)$$

for some $0 < \theta < 1$.

Also continuity of f_x at (a,b) implies that

$$\lim_{(h,k)\to(0,0)} f_x(a+\theta h, b+k) = f_x(a,b)$$

Therefore for some $\phi(h,k)$ such that $\lim\limits_{(h,k)\to(0,0)} \phi(h,k) = 0$, we get

$$f_x(a+\theta h, b+k) = f_x(a,b) + \phi(h,k) \text{ - - - } (2)$$

Also we have,

$$\lim_{(h,k)\to(0,0)} \frac{f(a,b+k) - f(a,b)}{k} = f_y(a,b)$$

Therefore for some $\psi(k)$ such that $\lim\limits_{k\to 0} \psi(k) = 0$ we get,

$$\frac{f(a,b+k) - f(a,b)}{k} = f_y(a,b) + \psi(k)$$

Therefore,

$$f(a,b+k) - f(a,b) = kf_y(a,b) + k\psi(k) \text{ - - - } (3)$$

Substituting from (2) and (3) in (1) we get,

$$\delta f = (hf_x(a,b) + h\phi(h,k)) + (kf_y(a,b) + k\psi(k))$$

Therefore,

$$\delta f = hf_x(a,b) + kf_y(a,b) + h\phi(h,k)) + k\psi(k)$$

Hence, f is differentiable at (a,b).

39. **Show that the function** $f(x,y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y}, & \textbf{if } xy \neq 0 \\ x^2 \sin\dfrac{1}{x}, & \textbf{if } x \neq 0,\ y = 0 \\ y^2 \sin\dfrac{1}{y}, & \textbf{if } x = 0,\ y \neq 0 \\ 0, & \textbf{if } x = 0, y = 0 \end{cases}$

possesses both the first order partial derivatives which are not continuous at $(0,0)$ but it is not differentiable at $(0,0)$.

__Proof__

Here, $f(x,y) = \begin{cases} x^2 \sin\dfrac{1}{x} + y^2 \sin\dfrac{1}{y}, & \text{if } xy \neq 0 \text{ - - - } (1) \\ x^2 \sin\dfrac{1}{x}, & \text{if } x \neq 0,\ y = 0 \text{ - - - } (2) \\ y^2 \sin\dfrac{1}{y}, & \text{if } x = 0,\ y \neq 0 \text{ - - - } (3) \\ 0, & \text{if } x = 0, y = 0 \text{ - - - } (4) \end{cases}$

For the function $f(x,y)$. if $x \neq 0$ then case (1) and (2) are applicable and for $x = 0$ cases (3) and (4) are applicable. Therefore,

$$f_x(x,y) = \begin{cases} 2x \sin\dfrac{1}{x} - \cos\dfrac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Also, if $y \neq 0$ then case (1) and (3) are applicable and for $y = 0$ cases (2) and (4) are applicable. Therefore,

$$f_y(x,y) = \begin{cases} 2y \sin\dfrac{1}{y} - \cos\dfrac{1}{y}, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0 \end{cases}$$

If $y \to 0-$ then $\frac{1}{y} \to -\infty$ and if $y \to 0+$ then $\frac{1}{y} \to \infty$. This implies that $\lim\limits_{x\to 0} \cos\dfrac{1}{x}$ does not exist.

Therefore, $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist. Hence, $f_x(x,y)$ is not continuous at $(0,0)$. Similarly, $f_y(x,y)$ also is not continuous at $(0,0)$.

Finally let us show that f is differntiable at $(0,0)$.

If (x,y) changes from $(0,0)$ to (h,k) then corrsponding change in δf is given by

$$\delta f = f(h,k) - f(0,0)$$

Therefore

$$\delta f = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k} - 0$$

$$\therefore\ \delta f = h(0) + k(0) + h\left(h\sin\frac{1}{h}\right) + k\left(k\sin\frac{1}{k}\right)$$

Since

$$\lim_{(h,k)\to(0,0)} h\sin\frac{1}{h} = 0 \quad \text{and} \quad \lim_{(h,k)\to(0,0)} k\sin\frac{1}{k} = 0$$

it follows that f is differentiable at $(0,0)$.

40. Prove that $f(x,y) = \sqrt{|xy|}$ is not differentiable at $(0,0)$ but f_x and f_y exist at $(0,0)$.

Proof

We have,

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt{|h(0)|}-0}{h} = \lim_{h\to 0}\frac{0}{h} = 0$$

Therefore,

$$f_x(0,0) = 0$$

Also,

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0}\frac{\sqrt{|k(0)|}-0}{k} = \lim_{k\to 0}\frac{0}{k} = 0$$

Therefore,

$$f_y(0,0) = 0$$

Thus, both the partial derivatives of first order exist at $(0,0)$.

If possible suppose f is derivable at $(0,0)$.

Then,

$$f(h,k) - f(0,0) = Ah + Bk + h\phi + k\psi$$

where, $A = f_x(0,0) = 0$, $B = f_y(0,0) = 0$, $\lim_{(h,k)\to(0,0)} \phi = 0$ and $\lim_{(h,k)\to(0,0)} \psi = 0$

Therefore,

$$\sqrt{|hk|} = h\phi + k\psi$$

Taking $h = r\cos\theta$ and $k = r\sin\theta$ we get,

$$\sqrt{r^2|\cos\theta\sin\theta|} = r(\phi\cos\theta + \psi\sin\theta)$$

$$\therefore\ (|\cos\theta\sin\theta|)^{\frac{1}{2}} = \phi\cos\theta + \psi\sin\theta$$

As for arbitray θ, $r\to 0$ implies that $(h,k)\to(0,0)$, taking $r\to 0$ we get,

$$(\cos\theta\sin\theta)^{\frac{1}{2}} = 0$$

which is not possible for all θ. Therefore, f is not differentiable at $(0,0)$

41. **Show that the function** $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \textbf{if } x^2+y^2 \neq 0 \\ 0, & \textbf{if } x=y=0 \end{cases}$

is differntiable at the origin.

Proof

We have,

$$\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} h(0)\frac{h^2-0}{h^2+0} = \lim_{h\to 0} 0 = 0$$

Therefore,

$$f_x(0,0) = 0$$

Also,

$$\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} k(0)\frac{0-k^2}{0+k^2} = \lim_{k\to 0} 0 = 0$$

Therefore,

$$f_y(0,0) = 0$$

Thus, both the partial derivatives of first order exist at $(0,0)$.

Also, for $x^2+y^2 \neq 0$,

$$\begin{aligned}\frac{\partial}{\partial x}\left(xy\frac{x^2-y^2}{x^2+y^2}\right) &= y\frac{\partial}{\partial x}\left(\frac{x^3-xy^2}{x^2+y^2}\right)\\ &= y\left(\frac{(x^2+y^2)(3x^2-y^2)-(x^3-xy^2)(2x)}{(x^2+y^2)^2}\right)\\ &= y\left(\frac{x^4+4x^2y^2-y^4)}{(x^2+y^2)^2}\right)\end{aligned}$$

Now, we show that f_x is continuous at $(0,0)$.

$$\begin{aligned}|f_x(x,y)-f_x(0,0)| &= \left|y\left(\frac{x^4+4x^2y^2-y^4}{(x^2+y^2)^2}\right)-0\right|\\ &\leqslant \left|y\frac{x^4+4x^2y^2}{(x^2+y^2)^2}\right|\\ &= \left|x^2y\frac{x^2+4y^2}{(x^2+y^2)^2}\right|\\ &\leqslant \left|x^2y\frac{4x^2+4y^2}{(x^2+y^2)^2}\right|\\ &\leqslant 4\left|x^2y\frac{x^2+y^2}{(x^2+y^2)^2}\right|\\ \therefore\ |f_x(x,y)-f_x(0,0)| &\leqslant 4\left|\frac{x^2y}{x^2+y^2}\right|\end{aligned}$$

For $x = r\cos\theta$ and $y = r\sin\theta$ we get,

$$\begin{aligned}|f_x(x,y) - f_x(0,0)| &\leqslant 4\left|\frac{(r^2\cos^2\theta)(r\sin\theta)}{r^2\cos^2\theta + r^2\sin^2\theta}\right| \\ &\leqslant 4r|\cos^2\theta\sin\theta| \\ &\leqslant 4\sqrt{x^2+y^2} \text{ - - - } (1)\end{aligned}$$

For any $\epsilon > 0$ if we take $\delta = \dfrac{\epsilon}{4\sqrt{2}}$ then

$$\begin{aligned}|x-0| < \delta,\ |y-0| < \delta &\Rightarrow |x| < \frac{\epsilon}{4\sqrt{2}},\ |y| < \frac{\epsilon}{4\sqrt{2}} \\ &\Rightarrow x^2 + y^2 < \frac{\epsilon^2}{32} + \frac{\epsilon^2}{32} \\ &\Rightarrow 16(x^2+y^2) < \epsilon^2 \\ &\Rightarrow 4\sqrt{x^2+y^2} < \epsilon \\ &\Rightarrow |f_x(x,y) - f_x(0,0)| < \epsilon \quad \text{(From (1))}\end{aligned}$$

Therefore,

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

Hence, f_x is continuous at $(0,0)$. Also as $f_y(0,0)$ exists, a sufficient condition for differentiability is satisfied. Hence, f is differntiable at $(0,0)$.

42. **Partial Derivatives of Higher Orders**

If the first order partial derivatives of a function f exist at each point of some region and $f_x(x,y)$ and $f_y(x,y)$ are also differentiable partially then those derivatives are called Second Order Partial Derivatives and are denoted as shown below,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_x^2$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$$

Proof

43. **For the function**

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } (x,y)=)(0,0)\end{cases}$$ **show that $f_{xy}\neq f_{yx}$ at the origin.**

Proof

44.

Young's Theorem: [Without Proof]
If f_x and f_y are both differentiable at a point (a,b) of the domain of a function f, then

$$f_{xy}(a,b)=f_{yx}(a,b)$$

45.

Schwarz's Theorem: [Without Proof]
If f_y exists in a certain neighbourhood of a point (a,b) of the domain of a function f, and f_{yx} is continuous at (a,b), then $f_{xy}(a,b)$ exists and is equal to $f_{yx}(a,b)$

46.

Differentials of Higher Order
If $z=f(x,y)$ is a function of two independent variables such that for a positive integer n all the partial derivatives upto n^{th} order exist then the n^{th} order differential of z is given by

$$\mathbf{d^n z}=\left(\frac{\partial}{\partial \mathbf{x}}\mathbf{dx}+\frac{\partial}{\partial \mathbf{y}}\mathbf{dy}\right)^{\mathbf{n}}\mathbf{z}$$