T.Y.B.Sc. : Semester - V (CBCS)

US05CMTH24

Metric Spaces and Topological Spaces

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US05CMTH24- UNIT : III

1. Cluster point

Cluster Point:
Let (X, T) be a topological space and $A \subset X$. A point p in X is said to be a cluster point of A if every T-neighbourhood of p contains atleast one point of A other than p. i.e.

NOTE:
The definition implies that if p is a cluster point of A and N is a neighbourhood of p then

$$(A - \{p\}) \cap N \neq \emptyset$$

2. Find the set of cluster points of $(1, 2)$ in usual topology and discrete topology of $\mathbb{R}$

Answer:

$\mathcal{U}$-topology
Here, $(1, 2)$ is a subset of R with usual topology, $\mathcal{U}$-topology.

First we show that each point of $[1, 2]$ is a cluster point of $(1, 2)$

Let r be any positive number.

Now, open interval $(1, 1 + r)$, contains atleast one point of $(1, 2)$ other than 1. So 1 is a cluster point of $(1, 2)$. Also, $(2 - r, 2)$ contains atleast one point of $(1, 2)$ other than 2. So 2 is a cluster point of $(1, 2)$.

Also, for any $c \in (1, 2)$, the interval $(c - r, c + r)$ contains atleast one point of $(1, 2)$ other than c. So c is a cluster point of $(1, 2)$.

Thus, each point in $[1, 2]$ is a cluster point of $(1, 2)$

Finally, we show that no point out side $[1, 2]$ can be a cluster point of $(1, 2)$

Let $x \notin [1, 2]$. If $2 < x$ then we can choose some sufficiently small $\epsilon > 0$ so that

$$2 < x - \epsilon < x$$

Therefore,

$$(1, 2) \cap (x - \epsilon, x + \epsilon) = \emptyset$$

So, x cannot be a cluster point of $(1,2)$.

Similarly it can be shown that if $x < 1$ then also x cannot be a cluster point of $(1,2)$ the set of all cluster points of $(1,2)$ is $[1,2]$

$\mathcal{D}$-topology

For R, we have the discrete topology $\mathcal{D}$ defined as the collection of all the subsets of R

Therefore, every subset of R is a $\mathcal{D}$-open set.

Therefore for every real number p, every subset of R containing p is a $\mathcal{D}$-neighbourhood of p. Consequently $\{p\}$ is a $\mathcal{D}$-neighbourhood of p.

Now, let A be any subset of R.

Since, for any $p \in R$, a $\mathcal{D}$-neighbourhood $\{p\}$ of p cannot contian any point of A, possibly other than p, we conclude that any real number p cannot be a cluster point of A. Hence, the set cluster points of every subset of R is $\emptyset$.

Therefore, the set of cluster points of $(1,2)$ is also $\emptyset$.

3. Find the sets of cluster points of the following subsets of $\mathbb{R}$

(1) R **(2)** $\left\{\frac{1}{n}/n \in J^+\right\}$ **(3)** $\left\{-\frac{1}{n}/n \in J^+\right\}$
(4) the set J of all integers.

relative to

(i) $\mathcal{U}$-topology (ii) $\mathcal{I}$-topology (iii) $\mathcal{D}$-topology

$\mathcal{U}$-topology

(1) The set of cluster points of R is R

(2) The set of cluster points of $\left\{\frac{1}{n}/n \in J^+\right\}$ is $\{0\}$

(3) The set of cluster points of $\left\{-\frac{1}{n}/n \in J^+\right\}$ is $\{0\}$

(4) The set of cluster points of J is $\emptyset$

$\mathcal{I}$-topology

For R, we have the indiscrete topology $\mathcal{I}$ given by $\mathcal{I} = \{\emptyset, R\}$

Therefore, the only non-empty $\mathcal{I}$-open set is R.

Therefore for every real number p there is only one $\mathcal{I}$-neighbourhood of p that is R.

Now, let A is a subset of R with more than one elements. For any $r \in R$ we can say that the only neighbourhood R of r contains entire A.

Because there are more than one elements in A, the only neighbourhood R of r contains atleast one element of A other than r.

Therefore, every $r \in R$ is a cluster point of A. Hence, R is the set of cluster points of A.

Now, each of R, $\left\{\frac{1}{n}/n \in J^+\right\}$, $\left\{-\frac{1}{n}/n \in J^+\right\}$ and J is a non-empty subset of R with more than none elements. Hence, R is the set of cluster points of these sets.

$\mathcal{D}$-topology

For R, we have the discrete topology $\mathcal{D}$ defined as the collection of all the subsets of R

Therefore, every subset of R is a $\mathcal{D}$-open set.

Therefore for every real number p, every subset of R containing p is a $\mathcal{D}$-neighbourhood of p. Consequently $\{p\}$ is a $\mathcal{D}$-neighbourhood of p.

Now, let A be any subset of R.

Since, for any $p \in R$, a $\mathcal{D}$-neighbourhood $\{p\}$ of p cannot contian any point of A, possibly other than p, we conclude that any real number p cannot be a cluster point of A. Hence, the set cluster points of every subset of R is $\emptyset$.

Now, each of R, $\left\{\frac{1}{n}/n \in J^+\right\}$, $\left\{-\frac{1}{n}/n \in J^+\right\}$ and J is a non-empty subset of R. Hence, $\emptyset$ is the set of cluster points of these sets.

NOTE:

In the context of indiscrete topology $\mathcal{I}$ its worth mentioning the case of a singleton case. For any $p \in R$, consider the singleton subset $\{p\}$ of R. We can say that the only $\mathcal{I}$-neighbourhood R of p **CANNOT** contain any point of $\{p\}$ other than p, hence p cannot be a cluster point of $\{p\}$.

Now for every $r \in R - \{p\}$ clearly the $\mathcal{I}$-neighbourhood R of r contians member p of $\{p\}$ which is certainly differnt from r, hence every $r \in R - \{p\}$ is a cluster point of $\{p\}$.

4. Let $(X, \mathcal{T})$ be a topological space. Prove that if F is $\mathcal{T}$-closed subset of X and $p \in (X - F)$ then there is a $\mathcal{T}$-neighbourhood N of p such that $N \cap F = \emptyset$

Proof:

Here, F is $\mathcal{T}$-closed subset of X and $p \in (X - F)$

Therefore, p cannot be a cluster point of F as every closed set must contain all its cluster points.

Therefore, there must be atleast one T-neighbourhood N of p such that

$$N \cap F = \emptyset$$

> **5.** **Let $(X, \mathcal{T})$ be a topological space. Find the set of all the cluster points of the empty subset of X**

Answer:
As the empty set does not contain any element, no neighbourhood of any point of X can contain a point of the empty set. Therefore the empty set has no cluster point. Hence the empty set itself is its set of cluster points.

> **6.** **Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X and A' be the set of all cluster points of A. Prove that A is $\mathcal{T}$-closed iff $A' \subset A$**

Proof:
Suppose A is a T closed subset of X.
If $p \in A'$ then p is cluster point of A. Therefore, every neighbourhood of p contains atleast one point of A other than p.

If possible suppose $p \notin A$. Therefore $p \in X - A$.

Since A is T-closed, its complement $X - A$ is T-open.
As $p \in X - A$, there is a T-neighbourhood N of p such that

$$N \subset (X - A)$$

Therefore no point of A is contained in N, a neighbourhood of p.
This contradicts the fact that p is a cluster of A. Therefore our supposition $p \notin A$ is wrong.

Hence,

$$p \in A' \Rightarrow p \in A$$

. Therefore, if A is closed then

$$A' \subset A$$

Conversely suppose $A' \subset A$. Now to prove that A is T-closed we shall show that $X - A$ is T-open.
If $p \in X - A$ then $p \notin A$. Since $A' \subset A$ we have $p \notin A'$ also.
Therefore p is not a cluster point of A. So there exists a T-neighbourhood N of p such that

$$N \cap A = \emptyset$$

But then

$$N \subset X - A$$

Therefore $X - A$ is a neighbourhood of each of its points. This implies that $X - A$ is T-open.

Hence, A is a T-closed set.

7. Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. Prove that $A \cup A'$ is $\mathcal{T}$-closed

Proof:
Here (X, T) is a topological space and $A \subset X$. To prove that $A \cup A'$ is T-closed we shall prove that its complement $X - (A \cup A')$ is T-open.
By the DeMorgan's law, we have

$$X - (A \cup A') = (X - A) \cap (X - A')$$

Now,

$$\begin{aligned} p \in [X - (A \cup A')] &\Rightarrow p \in [(X - A) \cap (X - A')] \\ &\Rightarrow p \in (X - A) \ \text{ and } \ p \in (X - A') \\ &\Rightarrow p \notin A \ \text{ and } \ p \notin A' \end{aligned}$$

Since $p \notin A'$, it is not a cluster point of A. Therefore there is a T-open neighbourhood U of p which does not contain any point of A other than p. As $p \notin A$, U does not contain any point of A. Therefore,

$$U \subset X - A$$

Since, U is a T-open neighbourhood of p which does not contain any point of A, it follows that no point of U is a cluster point of A.

Thus, U contains no points of A and no points of A' Therefore,

$$U \subset X - A'$$

Therefore,

$$U \subset [(X - A) \cap (X - A')]$$

Hence,

$$U \subset [X - (A \cup A')]$$

Therefore $X - (A \cup A')$ contains a neighbourhood of each of its points. Consequently $X - (A \cup A')$ is T-open. Hence, $A \cup A'$ is T-closed.

8. Closure of a set

Closure of a set
Let (X, T) be a topological space and $A \subset X$. The smallest T-closed subset of X containing A is called the closure of A and it is generally denoted by A^-.

9. Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Then prove that $A^- = A \cup A'$.

Proof:

Here $(X, \mathcal{T})$ is a topological space and $A \subset X$.

Now, $A \cup A'$ is a T-closed subset of X that contains A. As the closure A^- is the smallest T-closed subset of X which contains A we have

$$A^- \subset (A \cup A') \text{ - - - - (i)}$$

Next we show that $(A \cup A') \subset A^-$. If $p \in (A \cup A')$ then $p \in A$ or $p \in A'$.

In case $p \in A$ we have $p \in A^-$ as $A \subset A^-$.

Now if $p \in A'$ then p is a cluster point of A. Therefore every neighbourhood of p contains atleast one point of A other than p. Since $A \subset A^-$ we can say that every neighbourhood of p contains atleast one point of A^- also.
Therefore p is a cluster point of A^-. As A^- is a T-closed set we have $p \in A^-$.

Thus, $p \in A' \Rightarrow p \in A^-$. Since $A \subset A^-$ and $A' \subset A^-$ we get,

$$(A \cup A') \subset A^- \text{ - - - - (ii)}$$

From (i) and (ii) it follows that,

$$A^- = A \cup A'$$

10. Determine which of the following subsets of $\mathbb{R}$ are

(i) $\mathcal{U}$-closed (ii) $\mathcal{D}$-closed (iii) $\mathcal{I}$-closed

(a) $\mathcal{R}$ (b) $\left\{\frac{1}{n}/n \in J^+\right\}$ (c) $\left\{-\frac{1}{n}/n \in J^+\right\}$ (d) the set J of all integers

$\mathcal{U}$-topology

(1) We have, $R' = R$. Hence R is $\mathcal{U}$-closed.

(2) The set of cluster points of $A = \left\{\frac{1}{n}/n \in J^+\right\}$ is $\{0\}$ Since, $0 \notin A$, A is not $\mathcal{U}$-closed.

(3) The set of cluster points of $A = \left\{-\frac{1}{n}/n \in J^+\right\}$ is $\{0\}$ Since, $0 \notin A$, A is not $\mathcal{U}$-closed.

(4) The set of cluster points of J is $\emptyset$. Hence, J is not $\mathcal{U}$-closed.

$\mathcal{I}$-topology

For R, we have the indiscrete topology $\mathcal{I}$ given by $\mathcal{I} = \{\emptyset, R\}$

Therefore, the only non-empty $\mathcal{I}$-open set is R.

Therefore for every real number p there is only one $\mathcal{I}$-neighbourhood of p that is R.

Now, let A is a subset of R with more than one elements. For any $r \in R$ we can say that the only neighbourhood R of r contains entire A.

Because there are more than one elements in A, the only neighbourhood R of r contains atleast one element of A other than r.

Therefore, every $r \in R$ is a cluster point of A. Hence, R is the set of cluster points of A.

Now, each of R, $\left\{\frac{1}{n}/n \in J^{+}\right\}$, $\left\{-\frac{1}{n}/n \in J^{+}\right\}$ and J is a non-empty subset of R with more than none elements. Hence, R is the set of cluster points of these sets.

Hence, R is $\mathcal{I}$-closed and rest of these sets are not $\mathcal{I}$-closed as none them contains all their cluster points.

$\mathcal{D}$-topology

For R, we have the discrete topology $\mathcal{D}$ defined as the collection of all the subsets of R

Therefore, every subset of R is a $\mathcal{D}$-open set.

Therefore for every real number p, every subset of R containing p is a $\mathcal{D}$-neighbourhood of p. Consequently $\{p\}$ is a $\mathcal{D}$-neighbourhood of p.

Now, let A be any subset of R.

Since, for any $p \in R$, a $\mathcal{D}$-neighbourhood $\{p\}$ of p cannot contian any point of A, possibly other than p, we conclude that any real number p cannot be a cluster point of A. Hence, the set cluster points of every subset of R is $\emptyset$.

Now, each of R, $\left\{\frac{1}{n}/n \in J^{+}\right\}$, $\left\{-\frac{1}{n}/n \in J^{+}\right\}$ and J is a non-empty subset of R.Therefore, $\emptyset$ is the set of cluster points of these sets. Hence, all these sets are $\mathcal{D}$-closed.

11. Find $\mathcal{U}$-closures of the sets $\mathbb{R}$ and $\emptyset$.

$R^{-} = R$ and $\emptyset = \emptyset$

12. Dense Set:

Dense Set:

Let (X, T) be s topological space. A subset A of X is said to be dense in (X, T) if

$$A^{-} = X$$

13. Interior Point

Interior Point
Let (X, T) be s topological space and $A \subset X$. A point $p \in X$ is said to be T-interior point of A if A is a T-neighbourhood of p.

14. Interior

Interior
Let (X, T) be s topological space and $A \subset X$. The set of all the T-interior points of A is called the interior of A which is generally denoted by $IntA$.

15. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$. Prove the following
(i) $IntA \subset A$
(ii) $IntA$ **is a $\mathcal{T}$-open set**
(iii) A **is $\mathcal{T}$-open iff** $IntA = A$
(iv) $IntA$ **is the largest open subset of** A

(i) To Prove : $IntA \subset A$

If $p \in IntA$ then p is an interior point of A. Therefore these is some T-open subset, say G, of X such that

$$p \in G \subset A$$

Therfore,

$$p \in IntA \Rightarrow p \in A$$

Hence,

$$IntA \subset A$$

(ii) To Prove : $IntA$ is a $\mathcal{T}$-open set

If $p \in IntA$ then p is an interior point of A. Therefore these is some T-open subset, say G, of X such that

$$p \in G \subset A$$

Now, for each $x \in G$ we have $x \in G \subset A$ it follows that each point of G is an interior point of A.

Therefore,

$$G \subset IntA$$

Therefore, for each $p \in IntA$, there is some T-open subset of X such that

$$p \in G \subset IntA$$

This implies that $intA$ is a T-neighbourhood of each of its points. Hence $IntA$ is T-open.

(iii) To Prove : A is $\mathcal{T}$-open iff $IntA = A$

First we suppose that A is T-open. At (i) we have already proved that

$$IntA \subset A \text{ - - - } (1)$$

Now, as A is T-open for each $p \in A$ these is some T-open subset, say G, of X such that

$$p \in G \subset A$$

. Therefore each $p \in A$ is an interior point of A. therefore

$$p \in A \Rightarrow p \in IntA$$

Therefore,

$$A \subset IntA \text{ - - - } (2)$$

Form (1) and (2) it follows that $IntA = A$.

Thus, tf A is $\mathcal{T}$-open then $IntA = A$.

Conversely suppose, $IntA = A$.

If $p \in A$ then $p \in IntA$. Therefore there is some T-open subset G of X such that

$$p \in G \subset A$$

Therefore, A is a neighbourhood of each of its points. Hence A is open whenever $IntA = A$.

(iv) To Prove : $IntA$ is the largest open subset of A

At (i) we have already proved that $IntA \subset A$ and at (ii) we have proved that $IntA$ is a T-open set. Now let us prove that $IntA$ is the largest among all T-open subsets of A.

Let B be any T-open subset of A. Then B is a T-neighbourhood of each of its points.

Therefore, if $p \in B$ then there is some T-open set G such that

$$p \in G \subset B$$

Since, $B \subset A$, we have

$$p \in G \subset A$$

Therefore each $p \in B$ is an interior point of A, hence $p \in IntA$.

Thus,

$$p \in B \Rightarrow p \in IntA$$

Therefore,

$$B \subset IntA$$

Hence, $IntA$ is the largest open subset of A.

16. Continuous function

Continuous function :
Let (X, T) and (Y, ψ) be topological spaces. A function $f : X \to Y$ is called $T - \psi$-continuous if $f^{-1}(G)$ is T-open in X whenever G is ψ-open in Y

17. For any topologies $\mathcal{T}$ and Ψ of $\mathbb{R}$ show that the mapping $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = 2, \forall x \in \mathbb{R}$, is $\mathcal{T}$-Ψ continuous

Answer:
For any Ψ-open subset G of Y we have
$f^{-1}(G) = \begin{cases} X & ; \text{if } 2 \in G \\ \emptyset & ; \text{if } 2 \notin G \end{cases}$
Since, X and $\emptyset$ both are T-open, we can say that $f^{-1}(G)$ is T-open whenever G is Ψ-open. Hence, f is $T - \Psi$-continuous.

18. If $(X, \mathcal{T})$ and (Y, Ψ) are topological spaces and f is a mapping from X into Y then prove that the following statements are equivalent

(a) The mapping f is continuous
(b) The inverse image of f of every Ψ-closed set is T-closed set
(c) If $x \in X$ then inverse image of every Ψ-neighbourhood of $f(x)$ is a T-neighbourhood of x
(d) If $x \in X$ and N is a Ψ-neighbourhood of $f(x)$, then there is a T-neighbourhood M of x such that $f(M) \subset N$
(e) If $A \subset X$, then $f(A^-) \subset f(A)^-$

Here, $(X, \mathcal{T})$ and (Y, Ψ) are topological spaces. To prove the equivalence of the given statements we shall prove the following one by one.

$$(a) \Rightarrow (c),\ (c) \Rightarrow (d),\ (d) \Rightarrow (e),\ (e) \Rightarrow (b),\ (b) \Rightarrow (a)$$

To prove $(a) \Rightarrow (c)$
We assume that $f : X \to Y$ is $T - \Psi$-continuous on X.

Now, for any $x \in X$ we have $f(x) \in Y$. Let N be any Ψ-neighbourhood of $f(x)$. Therefore, for Ψ-open subset, say G, of Y we have

$$f(x) \in G \subset N$$

Therefore,

$$x \in f^{-1}(G) \subset f^{-1}(N)$$

We have, Since f is $T - \Psi$-continuous and G is Ψ-open in$Y$, the set $f^{-1}(G)$ is T-open in X.

Therefore, $f^{-1}(N)$ is a T-neighbourhood of x.

To prove $(c) \Rightarrow (d)$

We assume that, if $x \in X$ then inverse image of every Ψ-neighbourhood of $f(x)$ is a $\mathcal{T}$-neighbourhood of x. If we take $M = f^{-1}(N)$ then $f(M) \subset N$.

Thus, M is a T-neighbourhood of x such that $f(M) \subset N$

To prove $(d) \Rightarrow (e)$

We assume that, if $x \in X$ and N is a Ψ-neighbourhood of $f(x)$, then there is a $\mathcal{T}$-neighbourhood M of x such that $f(M) \subset N$.

Now, consider a subset A of X. As $A^{-} = A \cup A'$, we have,

$$f(A^{-}) = f(A \cup A') = f(A) \cup f(A')$$

Therefore, to show that $f(A^{-}) \subset f(A)^{-}$, it is sufficient to show that

$$f(A) \subset f(A)^{-} \quad \text{and} \quad f(A') \subset f(A)^{-}$$

Since, $f(A)^{-} = f(A) \cup (f(A))'$ it is clear that

$$f(A) \subset f(A)^{-} \text{ - - - } (1)$$

Next, to show that $f(A') \subset f(A)^{-}$, consider any $y \in f(A')$.

Clearly there is some $x \in A'$ such that $f(x) = y$. Let N be a Ψ-neighbourhood of $f(x)$. By our assumption there is a T-neighbourhood M of x such that $f(M) \subset N$.

As $x \in A'$, it is a cluster point of A. Therefore, the T-neighbourhood M of x contains atleast one point of A other than x. Therefore, $f(M)$ contains atlest one point of $f(A)$. As $f(M) \subset N$, it follows that N contains atleast one point of $f(A)$, which is either $f(x)$ or OTHER THAN $f(x)$.

As N is an arbitrary Ψ-neighbourhood of $f(x)$, it follows that $f(x)$ is either a member of $f(A)$ or a cluster point of $f(A)$. Therefore,

$$f(x) \in f(A) \cup f(A)'$$

Therefore,

$$f(x) \in f(A)^{-}$$

As $y = f(x)$, we have

$$y \in f(A') \Rightarrow y \in f(A)^{-}$$

Therefore,

$$f(A') \subset f(A)^{-} \text{ - - - } (2)$$

From (1) and (2) it follows that,

$$f(A) \cup f(A') \subset f(A)^{-}$$

Therefore,

$$f(A^-) \subset f(A)^-$$

To prove $(e) \Rightarrow (b)$

We assume that if $A \subset X$ then $f(A^-) \subset f(A)^-$

Let F be a Ψ-closed subset of Y. Therefore, $F^- = F$.

We shall show that the inverse image $f^{-1}(F)$ contains all its cluster points.

If p is a cluster point of $f^{-1}(F)$ then $p \in (f^{-1}(F))^-$.
Therefore

$$f(p) \in f\left[(f^{-1}(F))^-\right] \text{ - - - - (i)}$$

Now from our assumption $f(A^-) \subset f(A)^-$ we get

$$f\left[(f^{-1}(F))^-\right] \subset \left[f\left(f^{-1}(F)\right)\right]^- \text{ - - - - (ii)}$$

From (i) and (ii) it follows that

$$f(p) \in \left[f\left(f^{-1}(F)\right)\right]^-$$

But

$$\left[f\left(f^{-1}(F)\right)\right]^- \subset F^- = F$$

Therefore

$$f(p) \in F$$

Hence

$$p \in f^{-1}(F)$$

Thus, $f^{-1}(F)$ contains all its cluster points.
Hence, $f^{-1}(F)$ is T-closed whenever F is Ψ-closed.

To prove $(b) \Rightarrow (a)$

We assume that $f^{-1}(F)$ is T-closed whenever F is a Ψ-closed set.

To show that $f : X \to Y$ is $T - \Psi$-continuous, consider a Ψ-open subset G of Y.

Therefore, $F = Y - G$ is a Ψ-closed subset of Y. By our assumption $f^{-1}(F)$ is T-closed.

Now,

$$\begin{aligned} f^{-1}(G) &= f^{-1}(Y - F) \\ &= f^{-1}(Y) - f^{-1}(F) \\ f^{-1}(G) &= X - f^{-1}(F) \end{aligned}$$

As $f^{-1}(F)$ is T-closed subset of X, the set $f^{-1}(G)$ is T-open subset of X.

Thus, $f^{-1}(G)$ is T-open subset of X whenever G is Ψ-open subset of Y.

Hence, f is $T - \Psi$-continuous.

19. Let $(X, \mathcal{T})$ and (Y, Ψ) be topological spaces and f be a mapping from X into Y. Prove that if $f(A^-) \subset f(A)^-$ for $A \subset X$, then the inverse image of f of every Ψ-closed set is $\mathcal{T}$-closed set.

Here, $(X, \mathcal{T})$ and (Y, Ψ) are topological spaces and f is a mapping from X into Y. We assume that if $A \subset X$ then $f(A^-) \subset f(A)^-$

Let F be a Ψ-closed subset of Y. We shall show that the inverese image $f^{-1}(F)$ contains all its cluster points.

If p is a cluster point of $f^{-1}(F)$ then $p \in \left(f^{-1}(F)\right)^-$.
Therefore

$$f(p) \in f\left[\left(f^{-1}(F)\right)^-\right] \text{ ---- (i)}$$

Now from our assumption $f(A^-) \subset f(A)^-$ we get

$$f\left[\left(f^{-1}(F)\right)^-\right] \subset \left[f\left(f^{-1}(F)\right)\right]^- \text{ ---- (ii)}$$

From (i) and (ii) it follows that

$$f(p) \in \left[f\left(f^{-1}(F)\right)\right]^-$$

But

$$f\left[f^{-1}(F)\right]^- \subset F^- = F$$

Therefore

$$f(p) \in F$$

Hence

$$p \in f^{-1}(F)$$

Thus, $f^{-1}(F)$ contains all its cluster points.
Hence, $f^{-1}(F)$ is T-closed whenever F is Ψ-closed.

20.

21.

22.

23.

24. Bicontinuous function

Bicontinuous function:
Let (X, T) and (Y, Ψ) be topological spaces. A function f is said to be Bicontinuous if it is a $T-\Psi$-continuous function and $f(G)$ is Ψ-open whenever G is T-open in X.

25. Homeomorphism

Homeomorphism:
Let (X, T) and (Y, Ψ) be topological spaces. A function $f : X \to Y$ is said to be a $T-\Psi$-homeomorphism from X onto Y if it is a bicontinuous function which is one-one and X onto Y.

26. Homeomorphic Topological Spaces

Two topological spaces (X, T) and (Y, Ψ) are said to be Homeomorphic if there is a function $f : X \to Y$ which is a $T-\Psi$-homeomorphism form X onto Y.

27. Topologically Equivalent Spaces

Topologically Equivalent Spaces:
Two topological spaces (X, T) and (Y, Ψ) are said to be topologically equivalent if they are homeomorphic to each other.

28. Topological Invariant Property

Topological Invariant Property:
A property of a topological space (X, T) is said to be a topological property if it is also possese by every topological spaces homeomorphic to (X, T).