T.Y.B.Sc. : Semester - V (CBCS)

US05CMTH24

Metric Spaces and Topological Spaces

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1. Connected and Disconnected Topological Spaces.

Connected and Disconnected Topological Spaces

A topological space (X, T) is said to be a disconnected if exist two non-empty subsets A and B of X with the following three properties

(1) $A \cup B = X$

(2) $A^- \cap B = \emptyset$

(3) $A \cap B^- = \emptyset$

If the space (X, T) is not disconnected then it is called Connected.

NOTE:

By the definition it follows that a topological space (X, T) is connected if it is IMPOSSIBLE to find non-empty subsets A and B which satisfy above three properties.

2. Prove that if a topological space (X, T) has a non-empty proper subset A that is both T-open and T-closed, then (X, T) is disconnected.

Proof:

Suppose, a topological space (X, T) has a non-empty proper subset A which is T-open and T-closed both.

Define, $B = X - A$. Clearly B is also a non-empty proper subset of X such that,

$$A \cap B = \emptyset \text{ - - - } (1)$$

and

$$A \cup B = X \text{ - - - } (2)$$

As A is a T-closed subset of X, we have $A^- = A$. Therefore from (1) we get,

$$A^- \cap B = \emptyset \text{ - - - } (3)$$

Also, A is T-open and $B = X - A$ implies that B is T-closed. Hence $B^- = B$,
Therefore from (1) we get,

$$A \cap B^- = \emptyset \text{ - - - } (4)$$

Thus we have two non-empty subsets of A and B of X with the properties at (2), (3) and (4).

Hence, (X, T) is a disconnected space.

3. **Prove that if (X, T) is disconnected then there is a nonempty proper subset of X that is both T-open and T-closed.**

Proof:
Let (X, T) be a disconnected space. Therefore there exist two non-empty subsets A and B of X such that,

$$A \cup B = X, \ A^- \cap B = \emptyset, \ A \cap B^- = \emptyset$$

Since, $A \subset A^-$ and $A^- \cap B = \emptyset$ it follows that

$$A \cap B = \emptyset$$

Therefore, A is a proper subset of X as A and B are non-empty subsets of X and $A = X - B$.

As $A \cap B^- = \emptyset$ and $A \cup B^- = X$, we have,

$$A = X - B^-$$

Therefore, A is a T-open subset of X as B^- is a T-closed subset of X.

Also, as $A^- \cap B = \emptyset$ and $A^- \cup B = X$ we have,

$$B = X - A^-$$

Therefore, B is a T-open subset of X as A^- is a T-closed subset of X.

Since $A = X - B$, it follows that A is T-closed also.

Thus, A is a non-empty proper subset of X which is T-open and T-closed both.

4. **Prove that a topological space $(X, \mathcal{T})$ is disconnected iff X has a non-empty proper subset that is both $\mathcal{T}$-open and $\mathcal{T}$-closed.**

Proof can be given by using the proofs of above two theorems.

5. **Prove that every indiscrete space is connected.**

Proof:
For any non-empty set X, the indiscrete topology is given by $I = \{\emptyset, X\}$

Therefore, there is no PROPER subset of X which is I-open as well as I-closed.

Hence every indiscrete topology is connected.

6. **Prove that discrete space that has more than one point is disconnected.**

Proof:
A set with more than one elements always has atleast one non-empty proper subset.

Therefore, if a non-empty set X has more than one elements than its discrete topology $\mathcal{D}$, which is the family of all subsets of X, contains atleast one proper subset of X.

If A is a non-empty proper subset of X then A and its complement $X - A$ both are $\mathcal{D}$-open.

Since, $X - A$ is $\mathcal{D}$-open A is $\mathcal{D}$-closed also.

Therefore X has a proper subset which is $\mathcal{D}$-open as well as $\mathcal{D}$-closed.

Hence the discrete space is disconnected.

7. **For $X = \{a, b, c\}$ consider the topology $T = \{X, \emptyset, \{a, b\}, \{c\}\}$. Is (X, T) connected?**

Answer:
Here, $T = \{X, \emptyset, \{a, b\}, \{c\}\}$ is a topology for X

The subsets $\{a, b\}$ of X is T-open.

Also, $\{c\}$ is T-open and $\{a, b\} = X - \{c\}$. Therefore, $\{a, b\}$ is T-closed also.

Thus, the proper subset $\{a, b\}$ of X is T-open as well as T-closed.

Hence X is disconnected relative to T.

8. **Bounded above subset of R**

Bounded above subset of R
A subset A of R is said to be bounded above if there exists some fixed $K \in R$ such that

$$x \leqslant K, \ \forall x \in A$$

9. **Bounded below subset of R**

Bounded below subset of R
A subset A of R is said to be bounded below if there exists some fixed $K \in R$ such that

$$K \leqslant x, \ \forall x \in A$$

10. Bounded subset of R

Bounded subset of R
A suset A of R is said to be bounded if there exists some fixed $K_1, K_2 \in R$ such that

$$K_1 \leqslant x \leqslant K_2, \ \forall\, x \in A$$

11. Least Upper Bound

Least Upper Bound
Let A be a bounded subset of R. A real number u is said to be the least upper bound of A if

(1) $x \leqslant u, \ \forall x \in A$ and

(2) if $u' < u$ then there exists some $y \in A$ such that $u' < y \leqslant u$.

The Least Upper Bound of a set is also known as the Supremum of A.

NOTE:
In other words we can say that the smallest member of the set of all the upper bounds of a bounded above subset A of R is called the Least Upper Bound of A.

12. Greatest Lower Bound

Greatest Lower Bound
Let A be a bounded subset of R. A real number l is said to be the greatest lower bound of A if

(1) $l \leqslant x, \ \forall x \in A$ and

(2) if $l < l'$ then there exists some $y \in A$ such that $l \leqslant y < l'$.

The Greatest Lower Bound of a set is also known as the Infimum of A.

NOTE:
In other words we can say that the greatest member of the set of all the lower bounds of a bounded below subset A of R is called the Greatest Lower Bound of A.

13. State the Least Upper Bound property of R

Least Upper Bound property of R
Every non-empty subset of R which is bounded above has the least upper bound in R.

14. **Prove that the space $(R, \mathcal{U})$ is connected.**

Proof:
Suppose that A is a non-empty proper subset of R which is $\mathcal{U}$-open and $\mathcal{U}$-closed both. Clearly the complement $R - A$ also is non-empty.

Take some $p_0 \in R - A$ and $q_0 \in A$. Clearly $p_0 \neq q_0$.

By the law of Trichotomy we have either $p_0 < q_0$ or $p_0 > q_0$.

CASE 1 : $p_0 < q_0$
Define,

$$M = \{q \in A \; / \; p_0 < q\}$$

As $p_0 < q_0$ and $q_0 \in A$ we have $q_0 \in M$. Also, p_0 is a lower bound of M.

Thus, M is a non-empty subset of R which is bounded below. By the Order Completeness of R, M must have greatest lower bound in R. Let r_0 be the greatest lower bound of M.

If N is a $\mathcal{U}$-neighbourhood of r_0 then there is some open interval (a, b) such that

$$r_0 \in (a, b) \subset N$$

Since, $r_0 < b$ and r_0 is the greatest lower bound of M there exists some $q \in M$ such that

$$r_0 \leqslant q < b$$

Therefore, $q \in N$. Thus every $\mathcal{U}$-neighbourhood of r_0 contains a point of M. Since $M \subset A$ it follows that every $\mathcal{U}$-neighbourhood of r_0 contains a point of A. Therefore, $r_0 \in A$ or r_0 is a cluster point of A, hence $r_0 \in A^-$. Since, A is $\mathcal{U}$-closed, we have $A^- = A$. Therefore,

$$r_0 \in A$$

As A is $\mathcal{U}$-open also, r_0 is an interior point of A. Therefore there is some open interval (a_0, b_0) such that

$$r_0 \in (a_0, b_0) \subset A$$

Clearly $(a_0, r_0) \subset A$. As r_0 is the greatest lower bound of M the open interval (a_0, r_0) cannot contain any point of M. Therefore we cannot have $p_0 < r_0$ because in that case infinitely many members of (a_0, r_0) will be in M. Hence we have

$$r_0 \leqslant p_0$$

Also p_0 is a lower bound of M and r_0 is the greatest lower bound of M. This implies that

$$p_0 \leqslant r_0$$

Thus we must have

$$p_0 = r_0$$

This is a contradiction as $p_0 \in R - A$ and $q_0 \in A$.

Therefore our supposition is wrong. Hence $A = R$ or $A = \emptyset$.

CASE 2 : $q_0 < p_0$

Define,

$$M = \{q \in A \;/\; q < p_0\}$$

As $q_0 < p_0$ and $q_0 \in A$ we have $q_0 \in M$. Also, p_0 is an upper bound of M.

Thus, M is a non-empty subset of R which is bounded above. By the Order Completeness of R, M must have least upper bound in R. Let r_0 be the least upper bound of M.

If N is a $\mathcal{U}$-neighbourhood of r_0 then there is some open interval (a, b) such that

$$r_0 \in (a, b) \subset N$$

Since, $a < r_0$ and r_0 is the least upper bound of M there exists some $q \in M$ such that

$$a < q \leqslant r_0$$

Therefore, $q \in N$. Thus every $\mathcal{U}$-neighbourhood of r_0 contains a point of M. Since $M \subset A$ it follows that every $\mathcal{U}$-neighbourhood of r_0 contains a point of A. Therefore, $r_0 \in A$ or r_0 is a cluster point of A, hence $r_0 \in A^-$. Since, A is $\mathcal{U}$-closed, we have $A^- = A$. Therefore,

$$r_0 \in A$$

As A is $\mathcal{U}$-open also, r_0 is an interior point of A. Therefore there is some open interval (a_0, b_0) such that

$$r_0 \in (a_0, b_0) \subset A$$

Clearly $(r_0, b_0) \subset A$. As r_0 is the least upper bound of M the open interval (r_0, b_0) cannot contain any point of M. Therefore we cannot have $r_0 < p_0$ because in that case infinitely many members of (r_0, b_0) will be in M. Hence we have

$$p_0 \leqslant r_0$$

Also p_0 is an upper bound of M and r_0 is the least upper bound of M. This implies that

$$r_0 \leqslant p_0$$

Thus we must have

$$p_0 = r_0$$

This is a contradiction as $p_0 \in R - A$ and $q_0 \in A$.

Therefore our supposition is wrong. Hence $A = R$ or $A = \emptyset$.

Thus, it is impossible to find a non-empty proper subset of R which is $\mathcal{U}$-closed and $\mathcal{U}$-open both.

Hence, $(R, \mathcal{U})$ is connected.

15. **Assuming that connectedness is a topological property prove that $(R,\mathcal{U})$ and $(R,\mathcal{G})$ are not homeomorphic where $\mathcal{U}$ is usual topology for R and $\mathcal{G}$ is defined as follows**
$G \in \mathcal{G}$ if either G empty or it is a nonempty subset of R such that for every $p \in G$ there is some $H = \{x \in R/a \leqslant x < b\}$ for $a < b$ such that $p \in H \subset G$.

Proof:
We know that for $a < b$ each half-closed half-open interval $[a, b)$ is $\mathcal{G}$-open in R.

Now, consider the $\mathcal{G}$-open subset $[0,1)$ of R. We can express $[0,1)$ as follows,

$$R - [0,1) = \left(\bigcup_{i=1}^{\infty} [-i, 0)\right) \bigcup \left(\bigcup_{i=1}^{\infty} [1, i)\right)$$

Therefore, $R - [0,1)$ is a union of $\mathcal{G}$-open sets, hence it is $\mathcal{G}$-open. Therefore, $[0,1)$ is $\mathcal{G}$-closed also.

Since $[0,1)$ is a non-empty proper subset of R which is $\mathcal{G}$-open and $\mathcal{G}$-closed both, the topological space $(R,\mathcal{G})$ is disconnected.

Now, $(R,\mathcal{U})$ is connected. As connctedness is a topological property it must be possessed by any topological space homeomorphic to $(R,\mathcal{U})$. As $(R,\mathcal{G})$ is not connected we conclude that, $(R,\mathcal{G})$ and $(R,\mathcal{U})$ are not homeomorphic.

16. **Prove that a continuous image of connected space is connected**

Proof:
Let (X,T) and (Y,ψ) be topological spaces and let $f: X \to Y$ be a $T\psi$-continuous mapping of X onto Y.

Let (X,T) be connected. If possible suppose (Y,ψ) is disconnected.

Then there is some non-empty proper subset G of Y which is ψ-open as well ψ-closed. As f is X onto Y, we have $f^{-1}(G)$ also a non-empty subset of X.

Now, as f is $T\psi$-continuous, $f^{-1}(G)$ is T-closed as well as T-open in X. This is not possible as X is connected.

There our supposition that (Y,ψ) is disconnected is wrong. Hence, (Y,ψ) is also connected.

17. **Relative topology and Subspace**

Relative topology and Subspace:

Let (X, T) be a topological space and Y be a non-empty subset of X. The T-relative topology for Y denoted by T_Y is the collection of subsets of Y given by

$$\{G \cap Y \ / \ G \in T\}$$

The topological space (Y, T_Y) is called a subspace of (X, T).

18. Open Set in a relative topology

Open Set in a relative topology:

Let (X, T) be a topological space and (Y, T_Y) be its subspace. Then a subset S of Y is said to be T_Y-open if and only if there exists some T-open subset G of X such that

$$S = G \cap Y$$

19. Show that a relative topology satisfies all the conditions for becoming a topological space

Proof:

Let (X, T) be a topological space and Y be a non-empty subset of X. The T-relative topology for Y denoted by T_Y is the collection of subsets of Y given by

$$T_Y = \{G \cap Y \ / \ G \in T\}$$

let us show that T_Y satisfies all the properties of a toplogical space.

We have $\emptyset, X \in T$. As $\emptyset \cap Y = \emptyset$ and $X \cap Y = Y$.

Hence,

$$\emptyset, Y \in T_Y \text{ - - - } (1)$$

Next consider an arbitrary collection $\{S_\alpha \ / \ \alpha \in \Lambda\}$ of of members of T_Y. Since each S_α is T_Y open there corresponds some T-open set G_α such that

$$S_\alpha = G_\alpha \cap Y$$

Now,

$$\bigcup_{\alpha \in \Lambda} S_\alpha = \bigcup_{\alpha \in \Lambda} (G_\alpha \cap Y) = \left(\bigcup_{\alpha \in \Lambda} G_\alpha \right) \cap Y$$

Since, each G_α is T-open, the union $\bigcup_{\alpha \in \Lambda} G_\alpha$ is T-open.

Therefore, $\left(\bigcup_{\alpha \in \Lambda} G_\alpha \right) \cap Y$ is T_Y open. Hence, $\bigcup_{\alpha \in \Lambda} S_\alpha$ is T_Y open.

Thus, arbitrary union of T_Y open sets is T_Y open - - - (2)

(iii) Finally, let $\{S_i \ / \ i = 1, 2, \ldots, n\}$ be a finite collection of members of T_Y. Since each S_i is T_Y open there corresponds some T-open set G_i such that

$$S_i = G_i \cap Y$$

Now,

$$\bigcap_{i=1}^{n} S_i = \bigcap_{i=1}^{n} (G_i \cap Y) = \left(\bigcap_{i=1}^{n} G_i\right) \cap Y$$

Since, each G_i is T-open, the finite intersection $\bigcap\limits_{i=1}^{n} G_i$ is T-open.

Therefore, $\left(\bigcap\limits_{i=1}^{n} G_i\right) \cap Y$ is T_Y open. Hence, $\bigcap\limits_{i=1}^{n} S_i$ is T_Y open.

Thus, finite intersection of T_Y open sets is T_Y open - - - (3)

From (1),(2) and (3) it follows that T_Y possesses all the three properties of a toplological space.

20. The space $(R, \mathcal{U})$ and $((0,1), \mathcal{U}_{(0,1)})$ are homeomorphic. [Without proof]

21. If I_1 and I_2 be any two open intervals then $(I_1, \mathcal{U}_{I_1})$ and $(I_2, \mathcal{U}_{I_2})$ are homeomorphic. [Without proof]

22. If I is any open interval then space $(R, \mathcal{U})$ and $(I, \mathcal{U}_I)$ are homeomorphic. [Without proof]

23. If I_1 and I_2 be any two closed intervals then $(I_1, \mathcal{U}_{I_1})$ and $(I_2, \mathcal{U}_{I_2})$ are homeomorphic. [Without proof]

24. Let (X, T) be a topological space and let Y be a subset of X. Prove that a subset S of Y is T_Y-closed iff there is a T-closed set F such that $S = F \cap Y$.

Proof:

Let S be a T_Y-closed subset of Y. Therefore $Y - S$ is a T_Y-open subset of Y.

Therefore, there exists some T-open subset G of X such that

$$Y - S = G \cap Y$$

Here $X - G$ is a T-closed subset of X. We shall show that $S = (X - G) \cap Y$.

Now,

$$\begin{aligned} p \in S &\iff p \notin Y - S \\ &\iff p \notin G \cap Y \qquad (\because Y - S = G \cap Y) \\ &\iff p \notin G \text{ and } p \in Y \\ &\iff p \in (X - G) \text{ and } p \in Y \\ &\iff p \in (X - G \cap Y \\ \therefore \quad p \in S &\iff p \in (X - G) \cap Y \\ \therefore \quad S &= (X - G) \cap Y \end{aligned}$$

As $X - G$ is a T-closed subset of X, taking $F = X - G$, we have

$$S = F \cap Y$$

where F is a T-closed set.

Conversely, suppose for some T-closed set F we have $S = F \cap Y$.

To show that S is T_Y closed we shall show that $Y - S$ is T_Y open.

Here, $X - F$ is T-open as F is T-closed. We shall show that $Y - S = (X - F) \cap Y$.

Now,

$$\begin{aligned} p \in Y - S &\iff p \notin S \text{ but } p \in Y \\ &\iff p \notin F \cap Y \text{ and } p \in Y \qquad (\because S = F \cap Y) \\ &\iff p \notin F \text{ and } p \in Y \\ &\iff p \in X - F \text{ and } p \in Y \\ \therefore \quad p \in Y - S &\iff p \in (X - F) \cap Y \\ \therefore \quad Y - S &= (X - F) \cap Y \end{aligned}$$

As $X - F$ is T-open $Y - S$ is T_Y-open. Hence S is T_Y-closed.

25. **Let (X, T) be a topological space and let Y be a subset of X. Prove that if the subsapce (Y, T_Y) is connected then so is the subspace (Y^-, T_{Y^-}).**

Proof:
For (X, T) and $Y \subset X$, let (Y, T_Y) be conncted.

If possible suppose (Y^-, T_{Y^-}) is disconnected. Then there is some subset, say A, of Y^- which is T_{Y^-}-closed and T_{Y^-}-open both. Therefore, there exist some T-open subset G and T-closed subset F of X such that

$$A = G \cap Y^- \quad \text{and} \quad A = F \cap Y^-$$

We note that $G \cap Y$ is T_Y-open and $F \cap Y$ is T_Y-closed. Now,

$$G \cap Y = G \cap (Y^- \cap Y) = (G \cap Y^-) \cap Y = A \cap Y$$

and

$$F \cap Y = F \cap (Y^- \cap Y) = (F \cap Y^-) \cap Y = A \cap Y$$

Therefore, we get,

$$G \cap Y = F \cap Y = A \cap Y$$

Let, $B = G \cap Y$. Then B is a T_Y-open and T_Y-closed subset of Y.

As (Y, T_Y) is connected B cannot be a non-empty proper subset of Y. Therefore we must have $B = \emptyset$ or $B = Y$.

CASE 1: $B = \emptyset$

Since $A \cap Y = B$, we have $A \cap Y = \emptyset$. As $A \subset Y^-$ we have

$$Y \subset Y^- - A$$

Now, A is T_{Y^-}-open implies that $Y^- - A$ is T_{Y^-}-closed. As Y^- is T-closed, $Y^- - A$ must be T-closed.

As Y^- is the smallest T-closed set containing Y, and $Y \subset Y^- - A$, we get

$$A = \emptyset$$

Which is a contradiction.

CASE 2: $B = Y$

Since $A \cap Y = B$, we have $A \cap Y = Y$. Therefore, $Y \subset A$.

As A is T_{Y^-}-closed and Y^- is T-closed, A is T_Y-closed.

Also, Y^- is the smallest T-closed set containing Y and A is a T-closed subset containing Y implies that, $Y^- \subset A$. Since, $A \subset Y^-$ we have

$$A = Y^-$$

Which is a contradiction.

Therefore, our supposition is wrong. Hence, (Y^-, T_{Y^-}) is connected.

26. Covering and Subcovering

Covering

A collection $\mathcal{B} = \{S_\alpha / \alpha \in \Lambda\}$ of subsets of a set X , where Λ is index set, is called a covering for X if

$$\bigcup_{\alpha \in \Lambda} S_\alpha = X$$

Sub-covering

If $\mathcal{B}_1$ and $\mathcal{B}_2$ both are covering of a set X and $\mathcal{B}_2 \subset \mathcal{B}_1$ then $\mathcal{B}_2$ is called a subcovering of $\mathcal{B}_1$.

27. Open Covering

Open Covering:

Let (X, T) be a toplogical space. A collection $\mathcal{B} = \{G_\alpha / \alpha \in \Lambda\}$ of T-open subsets of a set X , where Λ is index set, is called a T-open covering for X if

$$\bigcup_{\alpha \in \Lambda} G_\alpha = X$$

28. Compact Space

Compact Space:

A topological space (X, T) is said to be a Compact Space if every T-open covering of X has a finite subcovering.

29. Hausdorff Space

Hausdorff Space

A topological space (X, T) is said called a Hausdorff space or T_2 space if for every pair of distinct points p and q in X there exist some T-nbhds N_p and N_q of p and q respectively such that

$$N_p \cap N_q = \emptyset$$

30. Prove that the space $(R, \mathcal{U})$ is not compact and hence prove that no open interval is compact in its relativized $\mathcal{U}$ topology.

Proof:

Let $\mathcal{B} = \{(-n, n) \;/\; n \in J^+\}$ be a collection of $\mathcal{U}$-open intervals, which are $\mathcal{U}$-open subsets of R.

For any $p \in R$ there exists some positive integer N_p such that $|p| \leqslant N_p$. Therefore $p \in (-N_p, N_p)$, hence

$$\bigcup_{n=1}^{\infty} (-n, n) = R$$

Therefore, $\mathcal{B}$ is a $\mathcal{U}$-open convering of R.

Now, consider any finite collection $(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)$ of members of $\mathcal{B}$. Let $N = max\{n_1, n_2, \ldots, n_k\}$. Therefore $n_i \leqslant N$. Hence,

$$(-n_i, n_i) \subset (-N, N)$$

Therefore,

$$\bigcup_{i=1}^{k}(-n_i, n_i) \subset (-N, N)$$

Clearly $N \notin (-N, N)$. This implies that

$$\bigcup_{i=1}^{k}(-n_i, n_i) \neq R$$

Hence, the $\mathcal{U}$-open covering $\mathcal{B}$ of R does not have a finite subcovering. Hence $(R,\mathcal{U})$ cannot be compact.

Finally let I be an open interval. Therefore $(R,\mathcal{U})$ and $(I, \mathcal{U}_\mathcal{I})$ are homeomorphic to each other. Since compactness is a topological property which is not possessed by $(R,\mathcal{U})$, the subspace $(I, \mathcal{U}_\mathcal{I})$ also cannot possess this property. Hence, $(I, \mathcal{U}_\mathcal{I})$ is not connected.

31. **If $(Y, \mathcal{T}_Y)$ is a compact subspace of a Hausdorff space $(X, \mathcal{T})$, then prove that Y is $\mathcal{T}$ closed.**

Proof:

$(Y, \mathcal{T}_Y)$ is a compact subspace of a Hausdorff space $(X, \mathcal{T})$.

To prove that Y is T-closed, it is sufficient to prove that $X - Y$ is T-open.

Let $x \in X - Y$. Therefore, $x \notin Y$. For any $y \in Y$ we have $x \neq y$.

Since, $x \neq y$ and (X, T) is a Hausdorff space, there exist some T-open sets U_y and V_y such that

$$U_y \cap V_y = \emptyset$$

Corresponding to fixed $x \in X - Y$, and any $y \in Y$ we have $V_y \cap Y$ a T_Y-open subset of Y and

$$Y = \bigcup_{y \in Y}(V_y \cap Y)$$

Therefore, $\{V_y \cap Y \ / \ y \in Y\}$ is a T_Y-open covering of Y. Since (Y, T_Y) is compact, the T_Y-open covering of Y has a finite subcovering, say $\{V_{y_i} \cap Y \ / \ i = 1, 2, \ldots, n\}$ corresponding to some points y_i in Y.

Therefore, $Y = \bigcup\limits_{i=1}^{n}(V_{y_i} \cap Y)$. Hence,

$$Y \subset \bigcup_{i=1}^{n} V_{y_i}$$

Corresponging to each y_i there also corresponds a T-neighbourhood U_{y_i} such that $U_{y_i} \cap V_{y_i} = \emptyset$.

Let

$$G = \bigcup_{i=1}^{n} U_{y_i}$$

Clearly G is a T-neighbourhood of $x \in X - Y$ as each U_{y_i} is a T-neighbourhood of x.

Also as $U_{y_i} \cap V_{y_i} = \emptyset$ we have $G \cap \bigcup_{i=1}^{n} V_{y_i} = \emptyset$. Since $Y \subset \bigcup_{i=1}^{n} V_{y_i}$ we have $G \cap Y = \emptyset$. Therefore, $G \subset X - Y$. Therefore $X - Y$ is a T-neighbourhood of x.

As x is any point of $X - Y$, the set $X - Y$ is a T-neighourhood of each of its points. Hence $X - Y$ is a T-open set. Therefore, Y is T-closed.

32. If $(X, \mathcal{T})$ is compact and Y is a $\mathcal{T}$-closed subset of X, then prove that $(Y, \mathcal{T}_Y)$ is also compact.

__Proof:__
Let $(X, \mathcal{T})$ be a compact topological space and Y be a T-cloesed subset of X. Suppose $\mathcal{S} = \{S_\alpha \;/\; \alpha \in \lambda\}$ be a T_Y-open covering of Y. Therefore,

$$Y = \bigcup_{\alpha \in \Lambda S_\alpha}$$

Also as each S_α is T_Y-open, there corrsponds some T-open subset G_α of X such that $S_\alpha = G_\alpha \cap Y$. Therefore $Y = \bigcup_{\alpha \in \Lambda (G_\alpha \cap Y)}$. Therefore,

$$Y \subset \bigcup_{\alpha \in \Lambda} G_\alpha$$

Since, Y is T-closed $X - Y$ is T-open. Since, $X = (X - Y) \cap Y$ we have

$$X \subset (X - Y) \cup \left(\bigcup_{\alpha \in \Lambda} G_\alpha \right)$$

As the collection $(X - Y) \cup \{G_\alpha \;/\; \alpha \in \lambda\}$ is a T-open covering of X.

Since, (X, T) is compact, the T-open covering of X has a finite subcovering. The subset $X - Y$ is convered by only one T-open subset which is $X - Y$ itself. So the open subcovering must include $X - Y$. Suppose the T-open subcovering of X is

$$\{X - Y, G_1, G_2, \ldots, G_n\}$$

Then, We must have,

$$Y \subset \bigcup_{i=1}^{n} G_i$$

Therefore,

$$Y = \bigcup_{i=1}^{n} (G_i \cap Y)$$

For, $S_i = G_i \cap Y$ we get,

$$Y = \bigcup_{i=1}^{n} S_i$$

Therefore, $\{S_i \ / \ i = 1, 2, \ldots, n\}$ is a finite T_Y-open subcovering for Y. Hence (Y, T_Y) is compact.