B.Sc. (Semester - 5)
Subject: Physics
Course: US05CPHY21
Classical Mechanics

UNIT- II Moving Coordinate Systems and Motion of a Rigid Body

Introduction:

In many physical problems, we employ moving coordinate systems or frame of reference. In such problems, we usually consider two frames, one fixed in the laboratory and the other fixed on the moving system.

The laboratory frame used for observation can be termed as fixed frame. A moving frame of reference, in general, can possess a translation or rotational velocity with respect to a fixed frame of reference. Sometimes, a moving frame may possess both translational and rotational velocities relative to a fixed frame of reference.

A frame of reference moving with a constant velocity relative to a fixed frame is called an *inertial frame* of reference.

A frame of reference is accelerated relative to a fixed frame is called *non-inertial* frame of reference.

It is well known that rotational motion is always an accelerated motion. Hence, all frame of reference that are rotating relative to a fixed frame of reference are the non-inertial frame of reference.

The motion of a particle moving on the earth is usually described with reference to a frame fixed to the surface or to the center of the earth. This frame is really a non-inertial frame because of the earth's rotational motion. In the analysis of the motion of a rigid body, rotating coordinate systems are found to be useful.

Co-Ordinate Systems with Relative Translational Motion:

Let us consider two coordinate systems $O'(x', y', z')$ fixed in space and $O(x, y, z)$ moving with translational velocity with respect to first frame as shown in fig. 2.1

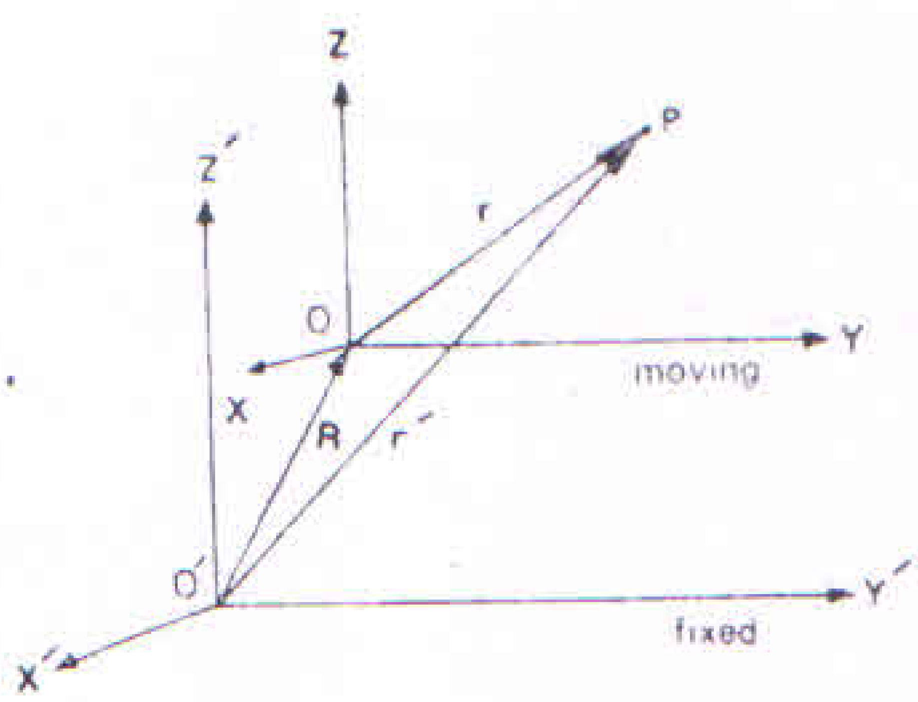


Fig: 2.1

Let $\vec{R}$ be the position vector of the point O with respect to the point O' at a certain instant t.

Let P be the point whose position vectors with respect to O and O' are $\vec{r}$ and $\vec{r}'$ respectively. From above figure,

$$\vec{r}' = \vec{R} + \vec{r} \qquad \text{... (2.1)}$$

Differentiation with respect to time gives,

$$\dot{\vec{r}}' = \dot{\vec{R}} + \dot{\vec{r}} \qquad \text{... (2.2)}$$

Thus, the velocity $\dot{\vec{r}}'$ of a particle at the point P in fixed frame is the velocity addition of $\dot{\vec{R}}$ and $\dot{\vec{r}}$. The corresponding accelerations are given by

$$\ddot{\vec{r}}' = \ddot{\vec{R}} + \ddot{\vec{r}} \qquad \text{... (2.3)}$$

The equation of motion of the particle at the point P in the fixed frame is

$$m\ddot{\vec{r}}' = \vec{F} \qquad \text{... (2.4)}$$

Where, $\vec{F}$ is the total external force acting on the particle P. The equation of motion of the particle at the point P in the moving frame of reference is

$$m\ddot{\vec{r}} = m\ddot{\vec{r}}' - m\ddot{\vec{R}}$$

$$\therefore\ m\ddot{\vec{r}} = \vec{F} - m\ddot{\vec{R}}$$

$$\therefore\ m\ddot{\vec{r}} = \vec{F}_{eff} \qquad \text{... (2.5)}$$

Thus, if the moving frame of reference has an acceleration $\ddot{\vec{R}}$, the effective force acting on the particle at point P is smaller than the actual force by an amount $m\ddot{\vec{R}}$. This reduction is due to acceleration $\ddot{\vec{R}}$ of the moving frame.

When, $\ddot{\vec{R}} = 0$, the equations of motion are identical in the two systems. In other words, Newton's laws of motion are valid in the two systems moving with a uniform relative velocity. This is known as the principle of *Newtonian relativity* or *Galilean invariance*. The form of equations remains the same in the two systems.

Rotating Coordinate Systems:

Let us suppose that the unprimed coordinate system $O(x, y, z)$ is rotating with angular velocity $\vec{\omega}$ about the axis passing through origin. The fixed and rotating systems have the common origin O. The fixed system is the primed system $O'(x', y', z')$.

The unit vectors $\hat{\imath}', \hat{\jmath}', \hat{k}'$ in the primed system are constant unit vectors while $\hat{\imath}, \hat{\jmath}, \hat{k}$ in the unprimed system are changing their directions along with the rotating axes.

The position vector of a particle at P can be written as

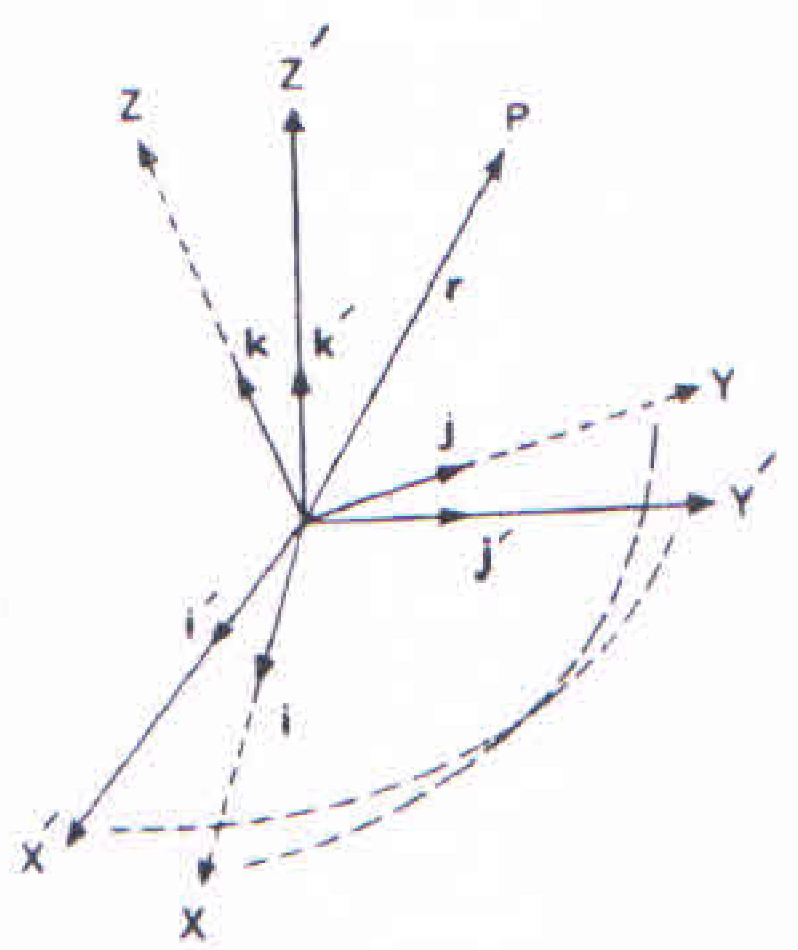


Fig : 2

$$\left.\begin{aligned}\vec{r} &= \hat{\imath}'x' + \hat{\jmath}'y' + \hat{k}'z' \\ and,\quad \vec{r} &= \hat{\imath}x + \hat{\jmath}y + \hat{k}z\end{aligned}\right\} \qquad \dots(2.6)$$

The transformation equations from unprimed to primed system are

$$\left.\begin{aligned}x' &= (\vec{r}\cdot\hat{\imath}') = \hat{\imath}\cdot\hat{\imath}'x + \hat{\jmath}\cdot\hat{\imath}'y + \hat{k}\cdot\hat{\imath}'z \\ y' &= (\vec{r}\cdot\hat{\jmath}') = \hat{\imath}\cdot\hat{\jmath}'x + \hat{\jmath}\cdot\hat{\jmath}'y + \hat{k}\cdot\hat{\jmath}'z \\ z' &= (\vec{r}\cdot\hat{k}') = \hat{\imath}\cdot\hat{k}'x + \hat{\jmath}\cdot\hat{k}'y + \hat{k}\cdot\hat{k}'z\end{aligned}\right\} \qquad \dots(2.7)$$

The inverse transformations can be written as the dot product of $\vec{r}$ with $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$.

The vector function $\vec{V} = \vec{V}(t)$ can be written as

$$\vec{V} = \hat{\imath}V_x + \hat{\jmath}V_y + \hat{k}V_z = \hat{\imath}'V_x' + \hat{\jmath}'V_y' + \hat{k}'V_z' \qquad \dots(2.8)$$

The time derivative will be different in two systems. In the primed or fixed system, we have

$$\left(\frac{d\vec{V}}{dt}\right)_{fix} = \hat{\imath}'\dot{V}_x' + \hat{\jmath}'\dot{V}_y' + \hat{k}'\dot{V}_z' \qquad \dots(2.9)$$

Since the unit vectors are constant vectors. In the unprimed or the rotating system, the unit vectors are changing in directions.

$$\therefore \left(\frac{d\vec{V}}{dt}\right)_{fix} = \hat{\imath}\dot{V}_x + \hat{\jmath}\dot{V}_y + \hat{k}\dot{V}_z + \frac{d\hat{\imath}}{dt}V_x + \frac{d\hat{\jmath}}{dt}V_y + \frac{d\hat{k}}{dt}V_z \qquad \dots(2.10)$$

The first three terms of above equation are the time derivatives of the vector in the rotating system when the unit vectors $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are treated as constant unit vectors.

$$\therefore \left(\frac{d\vec{V}}{dt}\right)_{rot} = \hat{\imath}\dot{V}_x + \hat{\jmath}\dot{V}_y + \hat{k}\dot{V}_z = \vec{V}_r \qquad \dots(2.11)$$

This is the velocity in the rotating system.

Now, the linear velocity $\vec{v}$ of the particle is given by

$$\vec{v} = \frac{d\vec{r}}{dt} = \vec{\omega}\times\vec{r} \qquad \dots(2.12)$$

This formula can be applied to unit vectors as a special case.

Since, $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are the unit vectors in a system rotating with angular velocity $\vec{\omega}$, we have

$$\frac{d\hat{\imath}}{dt} = \vec{\omega}\times\hat{\imath},\ \frac{d\hat{\jmath}}{dt} = \vec{\omega}\times\hat{\jmath}, \qquad and\ \frac{d\hat{k}}{dt} = \vec{\omega}\times\hat{k} \qquad \dots(2.13)$$

Hence, equation (2.10) reduced to

$$\left(\frac{d\vec{V}}{dt}\right)_{fix} = \left(\frac{d\vec{V}}{dt}\right)_{rot} + \vec{\omega}\times\vec{V} \qquad \dots(2.14)$$

Equation (2.14) gives the relation between the time derivatives in the fixed and rotating coordinate systems. This is the operator equation like

$$\left(\frac{d}{dt}\right)_{fix} = \left(\frac{d}{dt}\right)_{rot} + \vec{\omega}\times \qquad \dots(2.15)$$

Now, we operate on $\vec{\omega}$, we get

$$\left(\frac{d\vec{\omega}}{dt}\right)_{fix} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} + \vec{\omega} \times \vec{\omega}$$

$$\therefore \left(\frac{d\vec{\omega}}{dt}\right)_{fix} = \left(\frac{d\vec{\omega}}{dt}\right)_{rot} = \dot{\vec{\omega}} \qquad \text{... (2.16)}$$

This equation shows that angular acceleration $\dot{\vec{\omega}}$ is the same in the fixed and the rotating system.

The second derivative of $\vec{V}$ can be find in a similar manner. Now, let us denote

$$\left(\frac{d}{dt}\right)_{fix} = \frac{d'}{dt} \quad and \quad \left(\frac{d}{dt}\right)_{rot} = \frac{d}{dt} \qquad \text{... (2.17)}$$

Then, we can write

$$\frac{d'^2\vec{V}}{dt^2} = \frac{d'}{dt}\left(\frac{d'\vec{V}}{dt}\right) = \frac{d'}{dt}\left[\frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V}\right]$$

$$\therefore \frac{d'^2\vec{V}}{dt^2} = \left[\frac{d}{dt} + \vec{\omega} \times \quad\right]\left[\frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{V}\right]$$

$$\therefore \frac{d'^2\vec{V}}{dt^2} = \frac{d^2\vec{V}}{dt^2} + \frac{d\vec{\omega}}{dt} \times \vec{V} + \vec{\omega} \times \frac{d\vec{V}}{dt} + \vec{\omega} \times \left[\frac{d\vec{V}}{dt} + \vec{\omega} \times \frac{d\vec{V}}{dt}\right]$$

$$\therefore \frac{d'^2\vec{V}}{dt^2} = \frac{d^2\vec{V}}{dt^2} + 2\vec{\omega} \times \frac{d\vec{V}}{dt} + \vec{\omega} \times \vec{\omega} \times \vec{V} + \frac{d\vec{\omega}}{dt} \times \vec{V} \qquad \text{... (2.18)}$$

The above relations can be used to obtain expression for velocity and acceleration of the particle at P.

Now, consider the origin O of the rotating coordinate system is also moving with respect to O'. Hence, we have

$$\vec{r}' = \vec{R} + \vec{r}$$

$$\therefore \left(\frac{d\vec{r}'}{dt}\right)_{fix} = \left(\frac{d\vec{R}}{dt}\right)_{fix} + \left(\frac{d\vec{r}}{dt}\right)_{fix} \qquad \text{... (2.19)}$$

Thus, by using equations (2.15), (2.16) & (2.18), we get

$$\left(\frac{d\vec{r}'}{dt}\right)_{fix} = \left(\frac{d\vec{R}}{dt}\right)_{fix} + \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times \vec{r} \qquad \text{... (2.20)}$$

And,

$$\left(\frac{d^2\vec{r}'}{dt^2}\right)_{fix} = \left(\frac{d^2\vec{R}}{dt^2}\right)_{fix} + \left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} + 2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \frac{d\vec{\omega}}{dt} \times \vec{r} \qquad \text{... (2.21)}$$

In compact form equations (2.20) and (2.21) becomes,

$$\dot{\vec{r}}_f' = \dot{\vec{R}}_f + \dot{\vec{r}}_f + \vec{\omega} \times \vec{r} \qquad \text{... (2.22)}$$

And,

$$\ddot{\vec{r}}_f' = \ddot{\vec{R}}_f + \ddot{\vec{r}}_f + 2\vec{\omega} \times \dot{\vec{r}}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r} \qquad \text{... (2.23)}$$

Here,

$2\vec{\omega} \times \dot{\vec{r}}_r$ is called the *Coriolis acceleration*

$\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is called the *Centripetal acceleration*

The centripetal acceleration of the particle at P is directed towards the axis of rotation and is perpendicular to it. It has magnitude

$$|\vec{\omega} \times (\vec{\omega} \times \vec{r})| = \omega^2 r\, sin\theta = \frac{v^2}{r\, sin\theta} \qquad \dots (2.24)$$

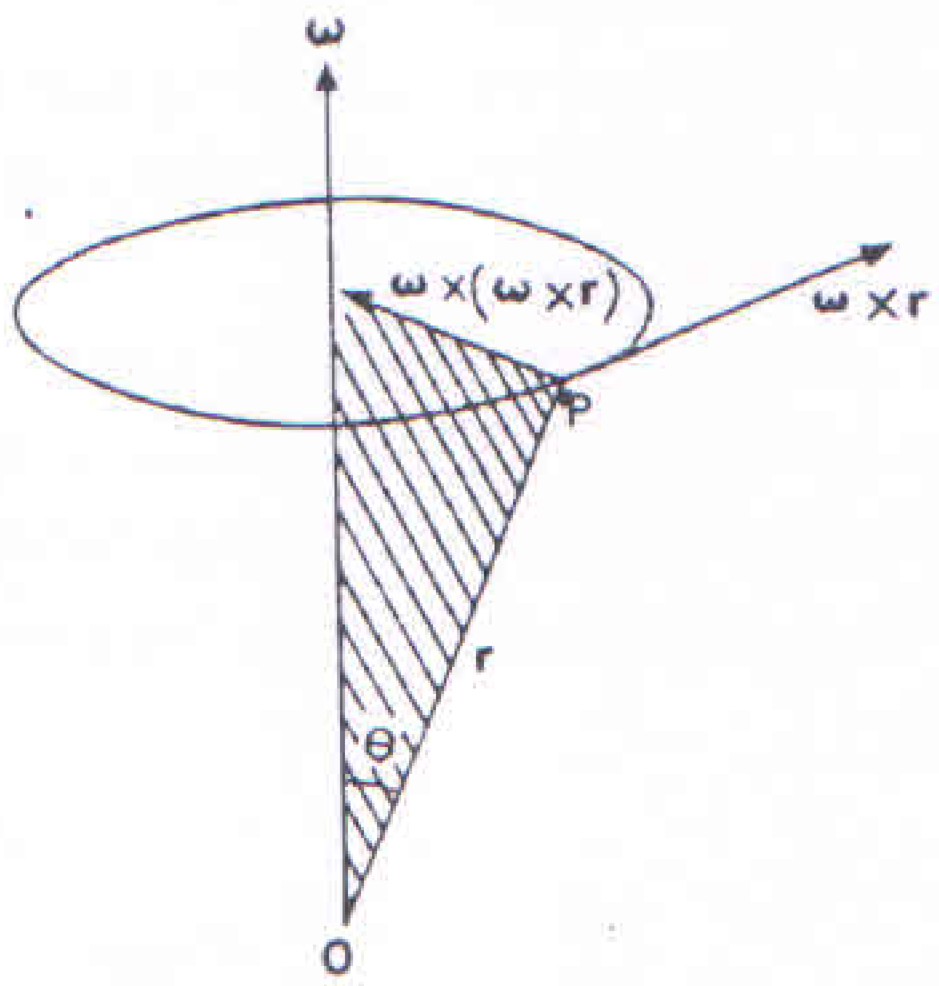


Fig:2.3

Where, $v = \omega r\, sin\theta$ is the speed of the particle when it rotates in a circle of radius $r\, sin\theta$ as shown in fig.2.3 . The Coriolis acceleration is present only when the particle has a velocity $\dot{\vec{r}}_r$ in the rotating frame.

The Coriolis Force:

Newton's second law of motion $\vec{F} = m\ddot{\vec{r}}$ is valid only in the inertial frames of references.

$$\therefore\ \vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix} \qquad \dots (2.25)$$

We wish to write the equation of motion of a particle in the rotating system.

Let the angular velocity of the rotating system be constant. Then, $\dot{\vec{\omega}} = 0$ and let the origins of the fixed and the moving systems coincide. Then, $\vec{R} = 0$ and $\vec{r} = \vec{r}'$.

Equation (2.21) becomes

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix} - 2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot} - m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

$$\therefore\ m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = \vec{F}_{eff} \qquad \dots (2.26)$$

Thus, the forces acting on the particle in the rotating frame are

- The real force $\vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix}$
- The centrifugal force $-m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$, and
- The Coriolis force $-2m\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot}$

The centrifugal and Coriolis forces are arising due to a non-inertial frame. These are added to the force term $\vec{F} = m\left(\frac{d^2\vec{r}}{dt^2}\right)$

$$\therefore\ \vec{F} + Non - inertial\ force = \vec{F}_{eff} \qquad ...(2.27)$$

Hence, the equation of motion of a particle in the rotating frame, the equation of motion resembles Newton's law i.e. $\vec{F} = m\ddot{\vec{r}}$.

Thus, for a satellite moving around the earth in the fixed system, only real force acting on it is gravitational force that produces the centripetal acceleration. An observer in the satellite, will experience only the effective force.

$$\therefore\ \vec{F}_{eff} = \vec{F} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad ...(2.28)$$

As the observer is at rest with respect to the resolving satellite. The centrifugal force is balance the gravitational force. An observer in a freely falling lift will not experience any force and will be in a weightless condition. An observer in a resolving satellite will also be in a weightless condition.

Motion on The Earth:

A particle of mass m situated on the surface of the earth will be acted upon by the gravitational force $m\vec{g}$. Other real force $\vec{F}$ such as an electrostatic or a magnetic force may also act on it.

Therefore, the equation of the particle is

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{fix} = \vec{F} + m\vec{g} \qquad ...(2.29)$$

The equation of motion of the same particle in the rotating system is

$$m\left(\frac{d^2\vec{r}}{dt^2}\right)_{rot} = \vec{F} + m[\vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r})] - 2\vec{\omega}\times\left(\frac{d\vec{r}}{dt}\right)_{rot} \qquad ...(2.30)$$

The second term of above equation represents the effective gravitational acceleration

$$\vec{g}_e = \vec{g} - \vec{\omega}\times(\vec{\omega}\times\vec{r}) \qquad ...(2.31)$$

The gravitational acceleration at any point will be this effective acceleration and it will be less than the acceleration due to the earth if it were not rotating. The term $-\vec{\omega}\times(\vec{\omega}\times\vec{r})$ is called the centrifugal acceleration as shown in fig.2.4. It always point radially outwards.

The centrifugal acceleration will be found to be zero at the pole. At the pole $\vec{\omega} \parallel \vec{r}$. It has the maximum value at the equator. The earth rotates in an anticlockwise sense about the North Pole with an angular velocity

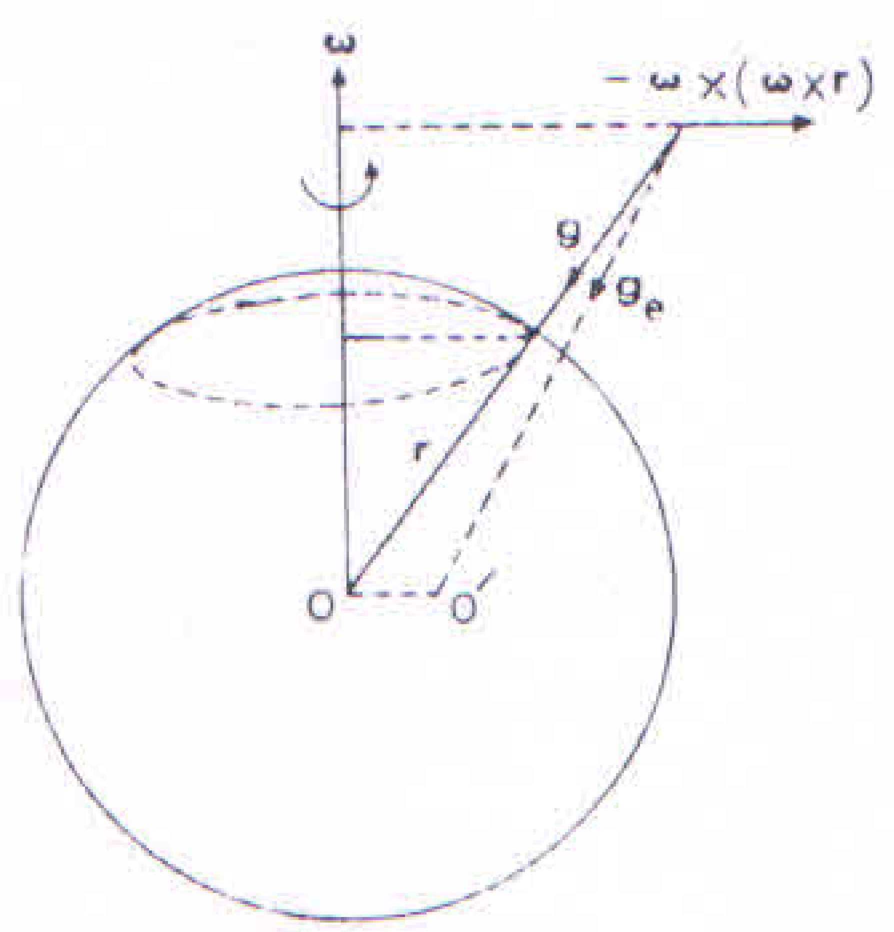


Fig.:2.4

$$\vec{\omega} = \frac{2\pi}{24 \times 60 \times 60} = 7.25 \times 10^{-5}\, rad/sec.$$

The radius of earth $r = 6.4 \times 10^6 m$, then the maximum centrifugal acceleration is

$$\omega^2 r = 3.4 \times 10^{-2}\, m/s^2$$

The third term of equation (2.30) is the Coriolis force that acts on a particle with a velocity $\vec{v}_r = \left(\frac{d\vec{r}}{dt}\right)_{rot}$ on the earth. The direction of the Coriolis force will be at right angles to the plane formed by $\vec{v}_r$ and $\vec{\omega}$.

Consider the effect of the Coriolis force on a particle situated at a point P and moving with a velocity $\vec{v}_r$ in a plane perpendicular to the axis of rotation of the earth and having latitude θas shown in figure 2.5

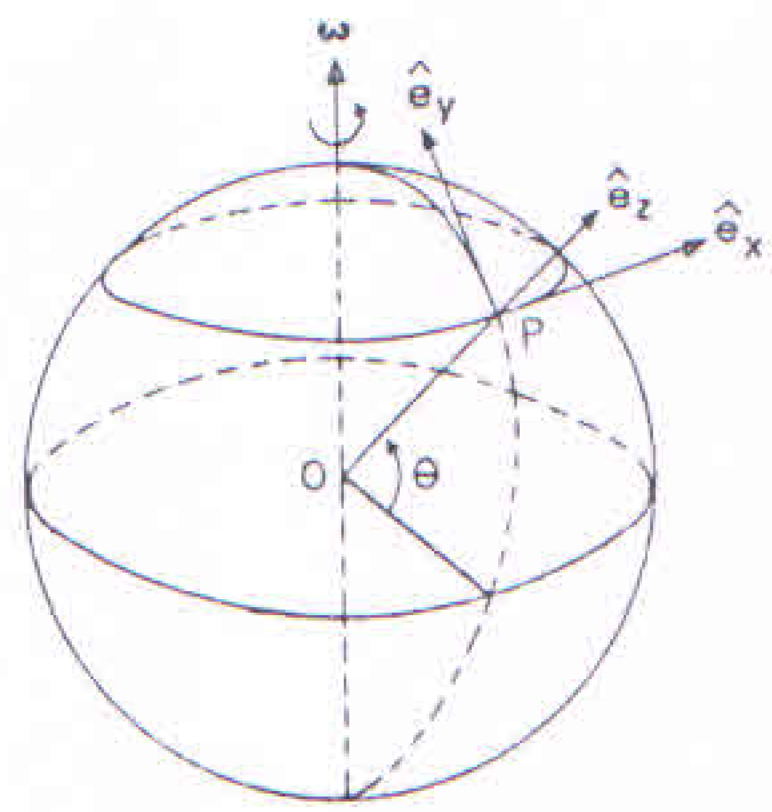


Fig.: 2.5

From figure it is clear that $\hat{e}_x$ and $\hat{e}_y$ axes from a horizontal plane at the point P while $\hat{e}_z$ - axis is vertical. The component of $\vec{\omega}$ along the vertical direction is $\omega_z \hat{e}_z = \hat{e}_z \omega \, sin\theta$ and , is directed upward in the northern hemisphere and downward in the southern hemisphere. Hence, the path of the particle will be deflected towards the right in the northern hemisphere and towards the left in the southern hemisphere due to the Coriolis acceleration.

The maximum magnitude of the Coriolis acceleration is at the north pole(Fig.2.6) or south pole and is given by

$$2\omega v_r = 1.5 \times 10^{-4} v_r$$

Where, v_r is the velocity in the horizontal plane.

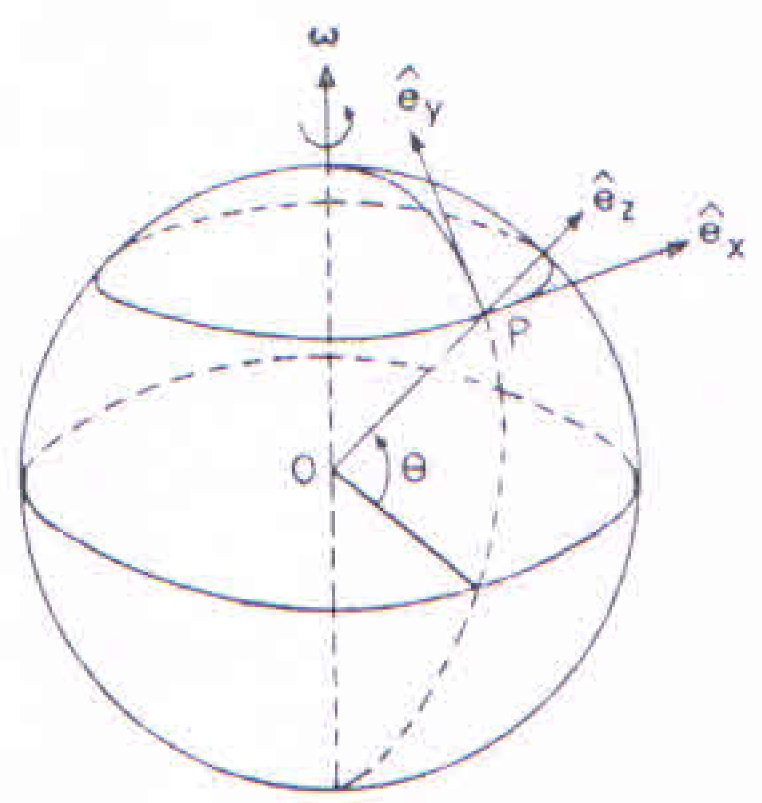


Fig: 2.6

The component of $\vec{\omega}$ along a vertical direction in a local coordinate system at the equator will be zero. Hence, the Coriolis force acting on a horizontally moving particle will be zero. The magnitude of the Coriolis acceleration is small, it plays an important role in many phenomena on the earth.

The effect of Coriolis acceleration is important in the flight of missiles, the velocity and the time of flight of which are considerably large.

Another terrestrial phenomena in which the Coriolis force plays an important role is the formation of cyclones. Whenever a low pressure region is formed, a mass of air will rush to this region from all directions. But, due to the Coriolis force, it will be deflected in the northern hemisphere as shown in fig.2.7

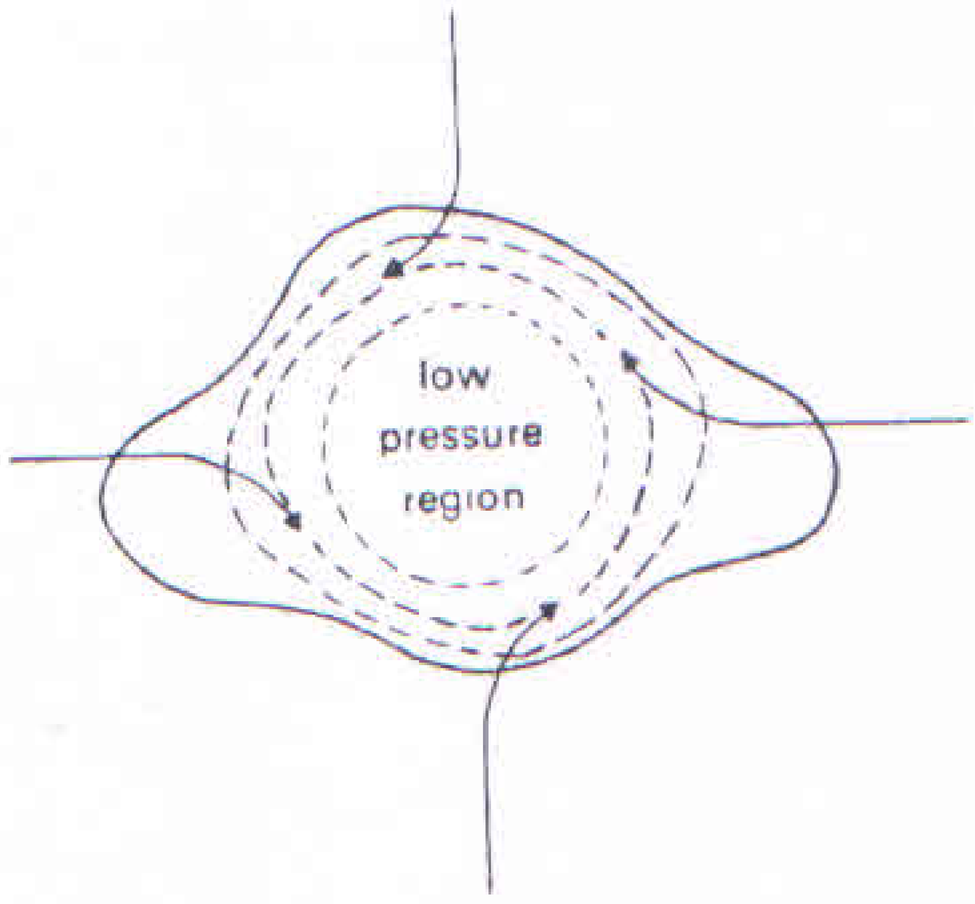


Fig: 2.7

Thus, in a cyclone the wind whirls in the anticlockwise sense in the northern hemisphere. The sense of rotation of wind will be clockwise in the southern hemisphere.

Effect of Coriolis Force on Freely Falling Particle:

Consider a particle is falling freely towards the earth. Let $'h'$ be the height of the particle from the earth be so small. Hence variation in $'g'$ can be neglected.

The acceleration of the particle is given by

$$\vec{a} = \vec{g} - 2\vec{\omega} \times \vec{v} \qquad \text{...(2.32)}$$

Where, $\vec{a}$ and $\vec{v}$ are measured in rotating frame with respect to the earth.

In the northern hemisphere, we have

$$\omega_x = 0\,, \omega_y = \omega\, cos\theta \quad and\; \omega_z = \omega\, sin\theta \qquad \text{...(2.33)}$$

The deflection produced by the Coriolis force is small. Hence, for a particle moving along $-\hat{e}_z$ direction, the component of forces along $\hat{e}_y$ and $\hat{e}_x$ are negligible.

$$\therefore\; \dot{x} \simeq 0, \dot{y} \simeq 0 \;\; and\; \dot{z} \simeq -gt \qquad \text{...(2.34)}$$

Hence, $\vec{\omega} \times \vec{v} = -\omega gt\; cos\theta \hat{e}_x$ lie along $x -$ direction.

As $\vec{g}$ is directed along the $-\hat{e}_z$ direction, the components of the acceleration of the particle are

$$a_x = \ddot{x} = 2\omega gt\; cos\theta \qquad \text{...(2.35)}$$

$$a_y = \ddot{y} = 0 \qquad \text{...(2.36)}$$

$$a_z = \ddot{z} = -g \qquad \text{...(2.37)}$$

The acceleration along $\hat{e}_x$ direction is due to the Coriolis force.

Now, integrating equations (2.35) and (2.37), we get

$$x = \frac{1}{3}\omega g t^3 \cos\theta \qquad \dots (2.38)$$

And, $$z = z_0 - \frac{1}{2} g t^2 \qquad \dots (2.39)$$

The initial conditions are

$$\left.\begin{matrix} z(0) = Z_0 \quad and \quad x(0) = 0 \\ \dot{x}(0) = 0 \ and \ \dot{z}(0) = 0 \end{matrix}\right\} \qquad \dots (2.40)$$

From equation (2.39), the time of fall from height is

$$t = \sqrt{\frac{2h}{g}} \qquad \dots (2.41)$$

Hence, the deflection of a particle towards the east is given by

$$x = \frac{1}{3}\omega g \cos\theta \left(\frac{2h}{g}\right)^{3/2} = \frac{1}{3}\omega \cos\theta \left(\frac{8h^3}{g}\right)^{1/2} \qquad \dots (2.42)$$

If a particle is dropped from a height of $100m$ from rest at latitude $\theta = 45^\circ$, it will be deflected by about $1.55 \times 10^{-2}m$. Here the effects of wind, viscosity etc. are neglected.

Euler's Theorem:

Statement: *Any general displacement of a rigid body, one point of which is fixed, is a rotation about some axis passing through the fixed point.*

As one point of the body is fixed, the body does not have any translation motion. According to the Euler's theorem, we can find out a single rotation about some axis.

Let us take the body-system of coordinates such that the origin of the system coincides with the fixed point. In the rotation motion, the position vector of any particle in the body does not change in its magnitude.

The theorem would be proved if we can find a straight line i.e. the axis of rotation such that the distance of the particles of the body from this straight line remains constant during the rotation.

Let A and B be the initial positions of two particles in the rigid body. A' and B' are the new positions after some arbitrary rotational displacement. Let O be the fixed point as shown in fig. 2.8

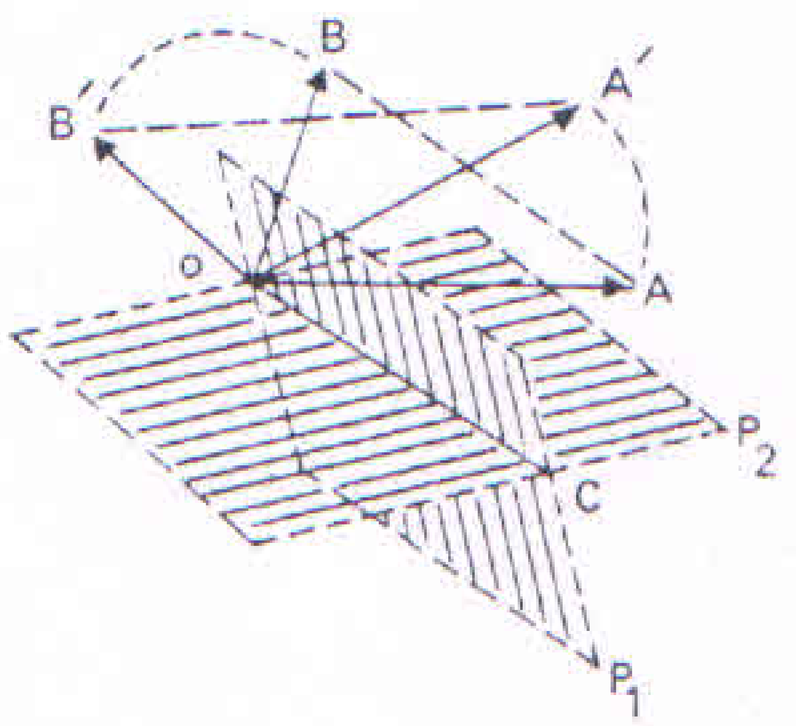


Fig.:2.8

The body was in the initial configuration OAB and its final configuration is $OA'B'$. Let us draw two planes $P_1$and P_2 perpendicular to the planes of the triangles OAA' and

OBB' respectively. The planes intersect each other along the straight line OC. Now, every point on the plane $P_1$is equidistant from points A and A'. Similarly, point $B$and B' are equidistant from plane P_2. The distances of the particles A and B from the straight line OC before and after rotation are equal. Hence, the straight line OC is the axis of rotation.

When the body is brought from the initial configuration OAB to its final configuration$OA'B'$, the line OC remains unchanged and the displacement is equivalent to a rotation about OC.

This configurations helps to understand another important theorem namely Chasles' theorem. It states that *the most general displacement of the rigid body is a translation plus a rotation about some axis*. The Chasles' theorem gives an idea to separate the motion of the rigid body into translation motion and rotational motion.

Angular Momentum and Kinetic Energy:

Consider a rigid body composed of n particles having masses m_a $(a = 1,2, \dots . n)$ and rotating with angular velocity $\vec{\omega}$.

Let one of the point in the body is fixed. Hence, translation motion is absent.

We shall find the expression for the angular momentum and the kinetic energy due to the rotation of the body.

The linear velocity $\vec{v}_a$ of the particle of mass m_a and position vector $\vec{r}_a$ with respect to the fixed point is given by

$$\vec{v}_a = \vec{\omega} \times \vec{r}_a \qquad \dots (2.43)$$

The total angular momentum $\vec{L}$ is the sum of angular momenta $\vec{l}_a$ of the individual particles and is given by

$$\vec{L} = \sum_{a=1}^{n} \vec{l}_a = \sum_a \vec{r}_a \times m_a \vec{v}_a$$

$$\therefore \ \vec{L} = \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)$$

Using vector triple product $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$ we get,

$$\therefore \ \vec{L} = \sum_a m_a \vec{\omega}(\vec{r}_a \cdot \vec{r}_a) - \sum_a m_a \vec{r}_a (\vec{r}_a \cdot \vec{\omega})$$

$$\therefore \ \vec{L} = \sum_a m_a r_a^2 \vec{\omega} - \sum_a m_a (\vec{r}_a \cdot \vec{\omega})\, \vec{r}_a \qquad \dots (2.44)$$

Now, the $x -$ component of the angular momentum is given by

$$L_x = \sum_a m_a r_a^2 \omega_x - \sum_a m_a [x_a \omega_x + y_a \omega_y + z_a \omega_z] x_a$$

$$\therefore \ L_x = \sum_a m_a (r_a^2 - x_a^2)\, \omega_x - \sum_a m_a x_a y_a \omega_y - \sum_a m_a x_a z_a \omega_z \qquad \dots (2.45)$$

Now, introduce the following symbols for the coefficients of ω_x , ω_y and ω_z.

$$I_{xx} = \sum_a m_a (r_a^2 - x_a^2) = \sum_a m_a (y_a^2 + z_a^2) \qquad \dots (2.46)$$

$$I_{yy} = \sum_a m_a(r_a^2 - y_a^2) = \sum_a m_a(z_a^2 + x_a^2) \quad \text{... (2.47)}$$

$$I_{zz} = \sum_a m_a(r_a^2 - z_a^2) = \sum_a m_a(x_a^2 + y_a^2) \quad \text{... (2.48)}$$

$$I_{xy} = -\sum_a m_a x_a y_a = I_{yx} \quad \text{... (2.49)}$$

$$I_{yz} = -\sum_a m_a y_a z_a = I_{zy} \quad \text{... (2.50)}$$

$$I_{zx} = -\sum_a m_a z_a x_a = I_{xz} \quad \text{... (2.49)}$$

With these substitutions, equation (2.45) and the other components of the angular momenta can be written as,

$$\left.\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}\right\} \quad \text{... (2.50)}$$

The quantities I_{xx} , I_{yy} and I_{zz} are called the *moment of inertia* about x , y and z axes respectively and I_{xy} , I_{yz} and I_{zx} are called the *product of inertia*.

If we denote x , y and z axes by 1 ,2 ,3 respectively, then reduce form of equation (2.50) is

$$L_i = \sum_i I_{ij}\omega_j \quad \text{... (2.51)}$$

Where, $i = 1,2,3$

The coefficients I_{ij} can always be calculated if the distribution of particles about the axes is known.

The kinetic energy of the rigid body can be calculated on similar ways.

We have,

$$T = \sum_a \frac{1}{2} m_a v_a^2 \quad \text{... (2.52)}$$

$$\therefore\ 2T = \sum_a m_a |v_a|^2 = \sum_a m_a (\vec{\omega} \times \vec{r}_a) \cdot (\vec{\omega} \times \vec{r}_a)$$

$$= \sum_a m_a \vec{\omega} \cdot [\vec{r}_a \times (\vec{\omega} \times \vec{r}_a)] = \vec{\omega} \cdot \sum_a m_a \vec{r}_a \times (\vec{\omega} \times \vec{r}_a)$$

$$\therefore\ 2T = \vec{\omega} \cdot \sum_a m_a \vec{r}_a \times \vec{v}_a = \vec{\omega} \cdot \vec{L}$$

$$\therefore\ T = \frac{1}{2} \vec{\omega} \cdot \vec{L} \quad \text{... (2.53)}$$

Above equation can also be written as,

$$\therefore\ T = \frac{1}{2} \sum_i \omega_i L_i = \frac{1}{2} \sum_{i,j} I_{ij} \omega_i \omega_j \quad \text{... (2.54)}$$

Above equation becomes

$$T = \frac{1}{2}\left[I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_x\right] \quad \text{... (2.55)}$$

If the body is rotating about the $z-$ axis with the angular velocity $\vec{\omega}$,

Then,

$$\omega_z = \omega \text{ and } \omega_x = \omega_y = 0$$

$\therefore$ Equation (2.55) becomes,

$$T = \frac{1}{2} I_{zz}\omega_z^2 = \frac{1}{2} I\omega^2$$

$$\therefore\ T = \frac{1}{2} I\omega^2 \qquad \ldots(2.56)$$

Where I is the moment of inertia of the body about the $z-$ axis.

The components of the angular momenta are,

$$L_x = I_{xz}\omega_z, \qquad L_y = I_{yz}\omega_z \ \ and \quad L_z = I_{zz}\omega_z \qquad \ldots(2.57)$$

This shows that the directions of the angular velocity and the angular momentum are different.

Let us consider a simple system of two particles having masses m_1 and m_2 connected by a rigid rod of negligible mass. The system is rotating with angular velocity $\vec{\omega}$ as shown in fig.2.9

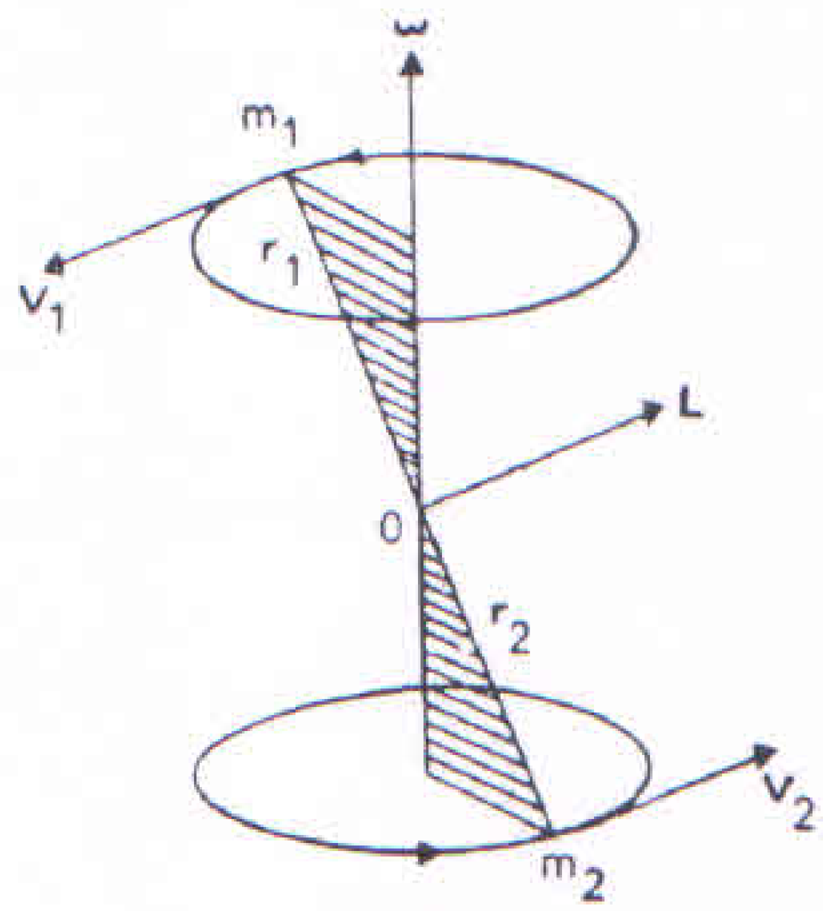


Fig:2.9

The angular momentum of the system is given by

$$\vec{L} = m_1\vec{r}_1 \times \vec{v}_1 + m_2\vec{r}_2 \times \vec{v}_2 \qquad \ldots(2.58)$$

The angular momentum $\vec{L}$ must be perpendicular to the connecting rod and is not parallel to the angular velocity $\vec{\omega}$.

As the particles rotates, the vector $\vec{L}$ which lies in the shaded plane as shown in fig:2.9. But it is perpendicular to the rod, also rotates and traces a cone with vertex at the point O.

Thus, $\dot{\vec{L}} \neq 0$ and according to the equation of motion

$$\dot{\vec{L}} = \vec{N} \qquad \ldots(2.59)$$

Hence, a torque $\vec{N}$ must be applied to maintain the rotation of the system about the given axis.

The Inertia Tensor:

The nine terms I_{ij} can be treated as the components of the moment of inertia of the rigid body. Each component is a scalar and has the dimension of $[M^1L^2]$. There are nine terms (3^2) of the component. Hence the moment of inertia is a tensor of rank two.

The moment of inertia tensor or the inertia tensor is denoted by $\overleftrightarrow{I}$. In matrix notation

$$\overleftrightarrow{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \qquad \ldots(2.60)$$

The moment of inertia tensor is symmetric. i.e.

$$I_{ij} = I_{ji}$$

Hence, it has only six independent.

If the matter is continuously distributed and the density function is $\rho = \rho(\vec{r})$ in the body, then moment of inertia is

$$I_{xx} = \int \rho(\vec{r})\,(r^2 - x^2)d\tau \qquad \ldots(2.61)$$

And the product of inertia is

$$I_{xy} = -\int \rho(\vec{r})\,xy\,d\tau \qquad \ldots(2.62)$$

We have

$$L_i = \sum_i I_{ij}\omega_j \qquad \text{Here, } i = j = 1,2,3,\ldots.$$

Above relation can be written as the dot product of the inertia tensor and the angular velocity vector.

$$\therefore \quad \vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \qquad \ldots.(2.63)$$

Now, the kinetic energy is

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} \qquad \ldots(2.64)$$

If the inertia tensor has only the diagonal elements then above equations (2.63) & (2.64) can be written as

$$L = I_i\omega_i \qquad \ldots(2.65)$$

And

$$T = \frac{1}{2}[I_i\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2] \qquad \ldots(2.66)$$

Thus, all the product of inertia are zero and the moment of inertia I_1,I_2 and I_3 are the non-zero elements.

Since, I_{ij} is symmetric by choosing the axes in the body such that the products of inertia are zero. These axes are called the *principal axes of inertia*.

- If the three component of the moment of inertia about the principal axes are equal i.e. $I_1 = I_2 = I_3$ then the body is called a *spherical top*.
- If two of the three components of the moment of inertia about the principal axes are equal to each other i.e. $I_1 = I_2 \neq I_3$, the body is called a *symmetrical top*.
- If all the components of the moment of inertia about the principal axes are different, i.e. $I_1 \neq I_2 \neq I_3$, then the body is known as an *asymmetric top*.

- A body for which $I_1 = I_2$ and $I_3 = 0$ is called a *rotator*. For example, diatomic molecule.

Euler's Equations of Motion:

Consider a rigid body, one point of which is fixed. Let $\vec{N}$ be the torque acting on it. Then the equation of rotational motion of the body in a fixed or inertial frame of reference is given by

$$\left(\frac{d\vec{L}}{dt}\right)_{fix} = \vec{N} \qquad \text{... (2.67)}$$

But, the operator is

$$\left(\frac{d}{dt}\right)_{fix} = \left(\frac{d}{dt}\right)_{rot} + \vec{\omega} \times \qquad \text{... (2.68)}$$

The rigid body is rotating with angular velocity $\vec{\omega}$. We shall fix a frame of reference. Hence, equation (2.67) becomes

$$\vec{N} = \left(\frac{d\vec{L}}{dt}\right)_{fix} = \left(\frac{d\vec{L}}{dt}\right)_{rot} + \vec{\omega} \times \vec{L} \qquad \text{... (2.69)}$$

But,

$$\left(\frac{d\vec{L}}{dt}\right)_{rot} = \left(\frac{d\vec{L}}{dt}\right)_{body} = \left[\frac{d}{dt}(\vec{I} \cdot \vec{\omega})\right]_{body}$$

$$\therefore \left(\frac{d\vec{L}}{dt}\right)_{rot} = \vec{I} \cdot \dot{\vec{\omega}} \qquad \text{... (2.70)}$$

But, $\vec{\omega}$ is same in the fixed and body frames of reference. Using equations (2.70) in (2.69)

$$\vec{N} = \vec{I} \cdot \dot{\vec{\omega}} + \vec{\omega} \times \vec{L} \qquad \text{... (2.71)}$$

If we orient the axes of the body frame of reference such that they coincide with the principal axes of the body. All the product of inertia will then vanish and the component form of equation (2.71) becomes

$$\left.\begin{aligned} N_1 &= I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \\ N_2 &= I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 \\ N_3 &= I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \end{aligned}\right\} \qquad \text{... (2.72)}$$

Equation (2.72) is known as the Euler's equations of the motion of a rigid body.

The solution of Euler's equations gives that how angular velocity changes with respect to the time and with respect to the principal axes. Hence, we should know relative change in $\vec{\omega}$ and $\vec{L}$.

If the torque acting on the rigid body is zero, then according to the law of conservation of angular momentum, the angular momentum $\vec{L}$ is a constant of motion. Hence, equation (2.69) becomes

$$\vec{\omega} \cdot \vec{L} = 0$$

$$\therefore \vec{\omega} \times (\vec{I} . \vec{\omega}) = 0 \qquad \text{... (2.73)}$$

Above equation will be true if $\vec{\omega} \parallel \vec{L}$.

$$\therefore \vec{L} = I\omega \qquad \text{... (2.74)}$$

Thus, in a torque-free motion, $\overleftrightarrow{I}\cdot\vec{\omega}$ is parallel to $\vec{\omega}$.

Now taking dot product of equation (2.71) with $\vec{\omega}$.

$$\therefore\ \vec{\omega}\cdot\vec{N} = \vec{\omega}\cdot\left[\overleftrightarrow{I}\cdot\dot{\vec{\omega}} + \vec{\omega}\times\vec{L}\right]$$

$$= \vec{\omega}\cdot\overleftrightarrow{I}\cdot\frac{d\vec{\omega}}{dt} \qquad \left(because\ \vec{\omega}\cdot\vec{\omega}\times\vec{L} = 0\right)$$

$$= \frac{d\vec{\omega}}{dt}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$$

$$\therefore\ \vec{\omega}\cdot\vec{N} = \frac{1}{2}\frac{d}{dt}\left(\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}\right) \qquad \ldots(2.75)$$

But, $\quad T = \frac{1}{2}\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}$

Hence, equation (2.75) becomes

$$\vec{\omega}\cdot\vec{N} = \frac{1}{2}\frac{d}{dt}\left(\vec{\omega}\cdot\overleftrightarrow{I}\cdot\vec{\omega}\right)$$

$$\therefore\quad \vec{\omega}\cdot\vec{N} = \frac{dT}{dt} \qquad \ldots(2.76)$$

Above equation gives the relation between the rate at which work is done by the torque and the rate of change of kinetic energy.

TORQUE FREE MOTION:

The motion of a free symmetric top is the simplest type of the motion of a rigid body in which the torque acting on it is known to be zero. A body is called a free symmetric top if $I_1 = I_2 \neq I_3$.

$\therefore$ We get the component of the torque

$$I_1\dot{\omega}_1 = (I_1 - I_3)\omega_2\omega_3 \qquad \ldots(2.77)$$

$$I_2\dot{\omega}_2 = (I_1 - I_3)\omega_3\omega_1 \qquad \ldots(2.78)$$

$$and\ \ I_3\dot{\omega}_3 = 0 \qquad \ldots(2.79)$$

Equation (2.79) shows that the component of the angular velocity ω_3 along the z −axis is constant.

$$\therefore\ \omega_3 = const \qquad \ldots(2.80)$$

Equations (2.77) & (2.78) can be written as

$$\dot{\omega}_1 = \left[\frac{(I_1 - I_3)\omega_3}{I_1}\right]\omega_2$$

$$\therefore\ \dot{\omega}_1 = \Omega\omega_2 \qquad \ldots(2.81)$$

$$Similarlly,\quad \dot{\omega}_2 = -\Omega\omega_1 \qquad \ldots(2.82)$$

$$where,\ \ \Omega = \frac{I_1 - I_3}{I_1}\omega_3 = const \qquad \ldots(2.83)$$

Now, differentiate equation (2.82) with respect to time ' t '

$$\therefore\ \ddot{\omega}_1 = \Omega\dot{\omega}_2$$

$$= \Omega^2\omega_1 \qquad (Using\ equation\ 2.82)$$

$$\therefore\ \ddot{\omega}_1 = -\Omega^2\omega_1 \qquad \ldots(2.84)$$

The solution of equation (2.84) is

$$\omega_1 = A\sin(\Omega t + \theta_0) \qquad \ldots(2.85)$$

Where, A and θ_0 are the constants.

Substituting this value of ω_1 in equation (2.82) and integrating it, we get

$$\omega_2 = A\cos(\Omega t + \theta_0) \qquad \text{... (2.86)}$$

Thus, components ω_1 and ω_2 of the angular velocity change in such a manner that their resultant ω_p is in the XY − plane and it rotates with angular frequency Ω.

$$\because\ \vec{\omega}_p = \hat{\imath}\omega_1 + \hat{\jmath}\omega_2 \qquad \text{... (2.87)}$$

The sense of Ω is the same as that of ω_3 if $I_1 > I_3$ and opposite to that of ω_3 if $I_1 < I_3$
The magnitude of this vector is given by

$$|\vec{\omega}_p| = A \qquad \text{... (2.88)}$$

Since, ω_3 is constant. Hence, the magnitude of the angular velocity $\vec{\omega}$ is also constant.

$$\therefore\ |\vec{\omega}| = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} = \sqrt{\omega_p^2 + \omega_3^2} = \sqrt{A^2 + \omega_3^2} = const. \qquad \text{... (2.89)}$$

Thus, the angular velocity vector $\vec{\omega}$ rotates about the body z − axis describe a core with the vertex at the origin as shown in fig.(2.10)

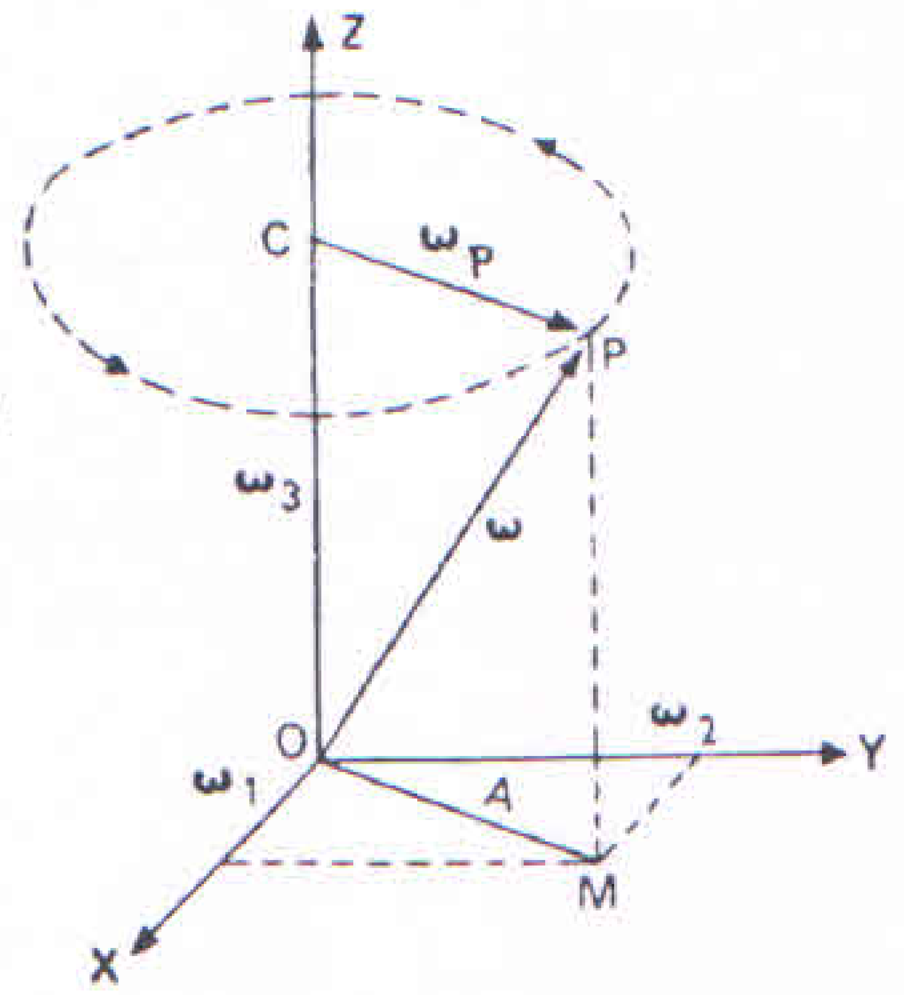


Fig: 2.10

This motion is called precession and the body is said to precess about the z − axis with precessional velocity Ω. The cone described by the angular velocity vector $\vec{\omega}$ is known as the body cone.

The half angle α_b of the body cone is given by

$$\tan\alpha_b = \frac{A}{\omega_3} \qquad \text{... (2.90)}$$

The constants ω_3 and A can be calculated in terms of the kinetic energy and angular momentum.

We know that,

$$T = \frac{1}{2}I_1A^2 + \frac{1}{2}I_3\omega_3^2 \qquad \text{... (2.91)}$$

$$and, \qquad L^2 = I_1^2A^2 + I_3^2\omega_3^2 \qquad \text{... (2.93)}$$

From equation (2.93), we have

$$A^2 = \frac{L^2 - I_3^2\omega_3^2}{I_1^2} \qquad \text{... (2.94)}$$

Substituting this value of A^2 in equation (2.91)

$$T = \frac{1}{2} I_1 \left[\frac{L^2 - I_3^2 \omega_3^2}{I_1^2} \right] + \frac{1}{2} I_3 \omega_3^2$$

$$\therefore 2T = \frac{L^2 - I_3^2 \omega_3^2}{I_1} + I_3 \omega_3^2$$

$$\therefore \ 2TI_1 = L^2 - I_3^2 \omega_3^2 + I_1 I_3 \omega_3^2$$

$$\therefore \ I_3^2 \omega_3^2 - I_1 I_3 \omega_3^2 = L^2 - 2TI_1$$

$$\therefore \ \omega_3^2 (I_3^2 - I_1 I_3) = L^2 - 2TI_1$$

$$\therefore \ \omega_3^2 = \frac{L^2 - 2TI_1}{I_3^2 - I_1 I_3}$$

$$\therefore \ \omega_3^2 = \frac{L^2 - 2TI_1}{I_3(I_3 - I_1)} \qquad \dots (2.95)$$

Putting this value of ω_3^2 in equation (2.93), we get

$$L^2 = I_1^2 A^2 + I_3^2 \left[\frac{L^2 - 2TI_1}{I_3(I_3 - I_1)} \right]$$

$$\therefore \ L^2 = I_1^2 A^2 + \frac{I_3 L^2 - 2TI_1 I_3}{I_3 - I_1}$$

$$\therefore \ I_1^2 A^2 = L^2 - \frac{I_3 L^2 - 2TI_1 I_3}{I_3 - I_1}$$

$$\therefore \ A^2 = \frac{I_3 L^2 - I_1 L^2 - I_3 L^2 + 2TI_1 I_3}{I_1^2 (I_3 - I_1)}$$

$$\therefore \ A^2 = -\frac{I_1 (L^2 - 2TI_3)}{I_1^2 (I_3 - I_1)}$$

$$\therefore \ A^2 = \frac{L^2 - 2I_3 T}{I_1 (I_1 - I_3)} \qquad \dots (2.96)$$

- In a torque free motion of a rigid body, the angular momentum of the body is a constant vector.
- Angular velocity $\vec{\omega}$ precesses about the z −axis which is also rotating with respect to the axes of the fixed frame of reference.

  Now, the angle between the two vectors $\vec{\omega}$ and $\vec{L}$ can be defined as

$$\cos \alpha_s = \frac{\vec{\omega} \cdot \vec{L}}{\omega L}$$

$$\therefore \ \cos \alpha_s = \frac{\vec{\omega} \cdot \vec{I} \cdot \vec{\omega}}{\omega L}$$

$$\therefore \ \cos \alpha_s = \frac{2T}{\omega L} \qquad \dots (2.97)$$

From above equation, an angle α_s is found to be constant. Thus, vector $\vec{\omega}$ rotates about vector $\vec{L}$ in a cone with half angle α_s. The position of vector $\vec{L}$ is fixed with respect to the space axes and hence the cone is called the *space cone*.

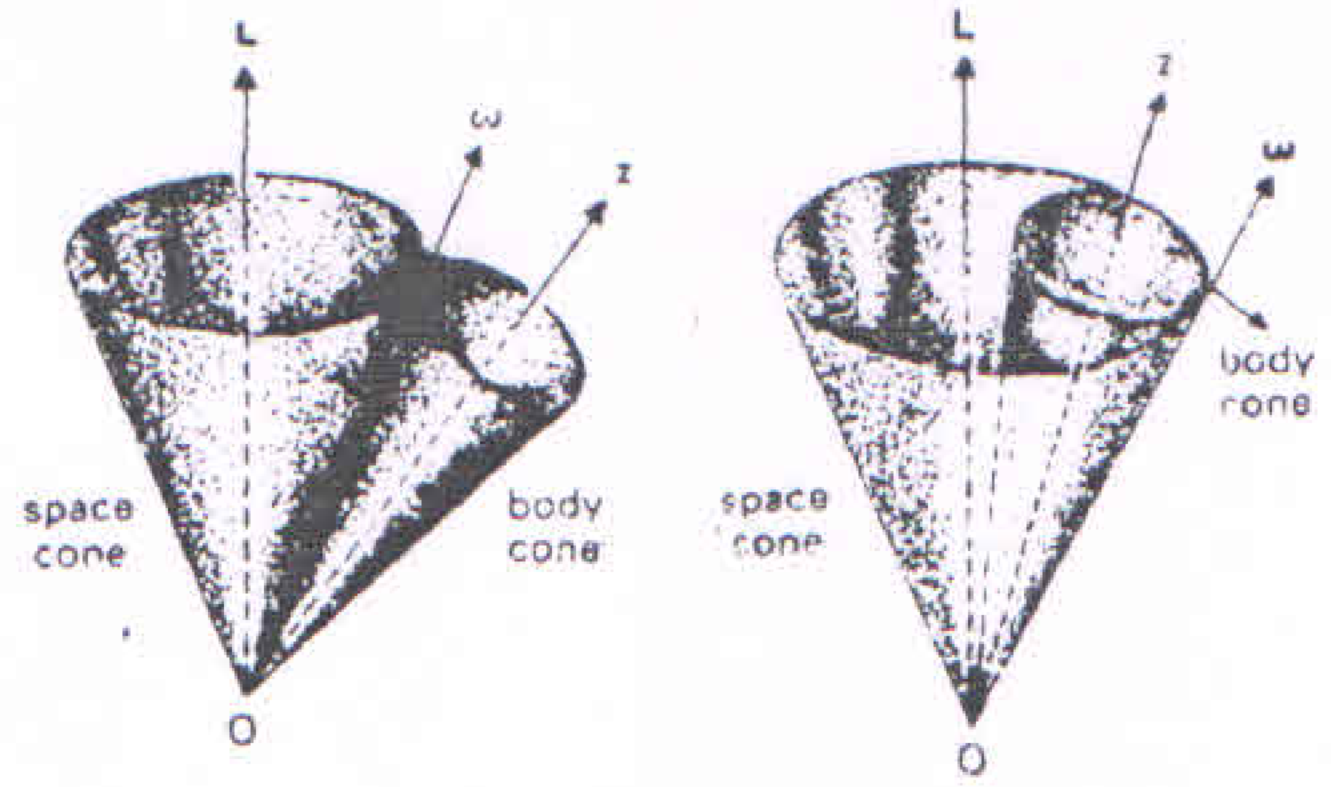


Fig: 2.11

The motion of the body cone rolling without slipping on the space cone is shown in fig. 2.11

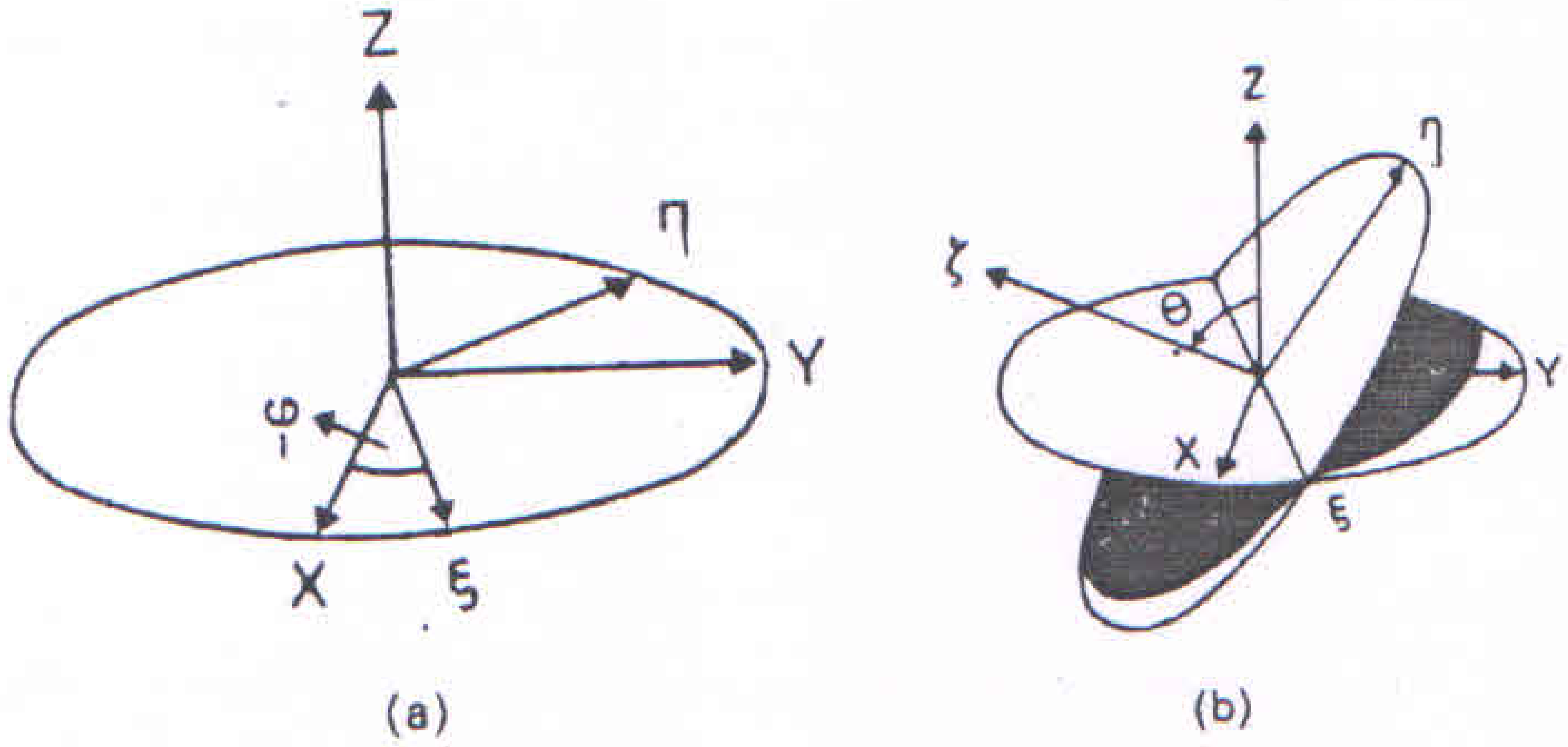
Euler's Angles:

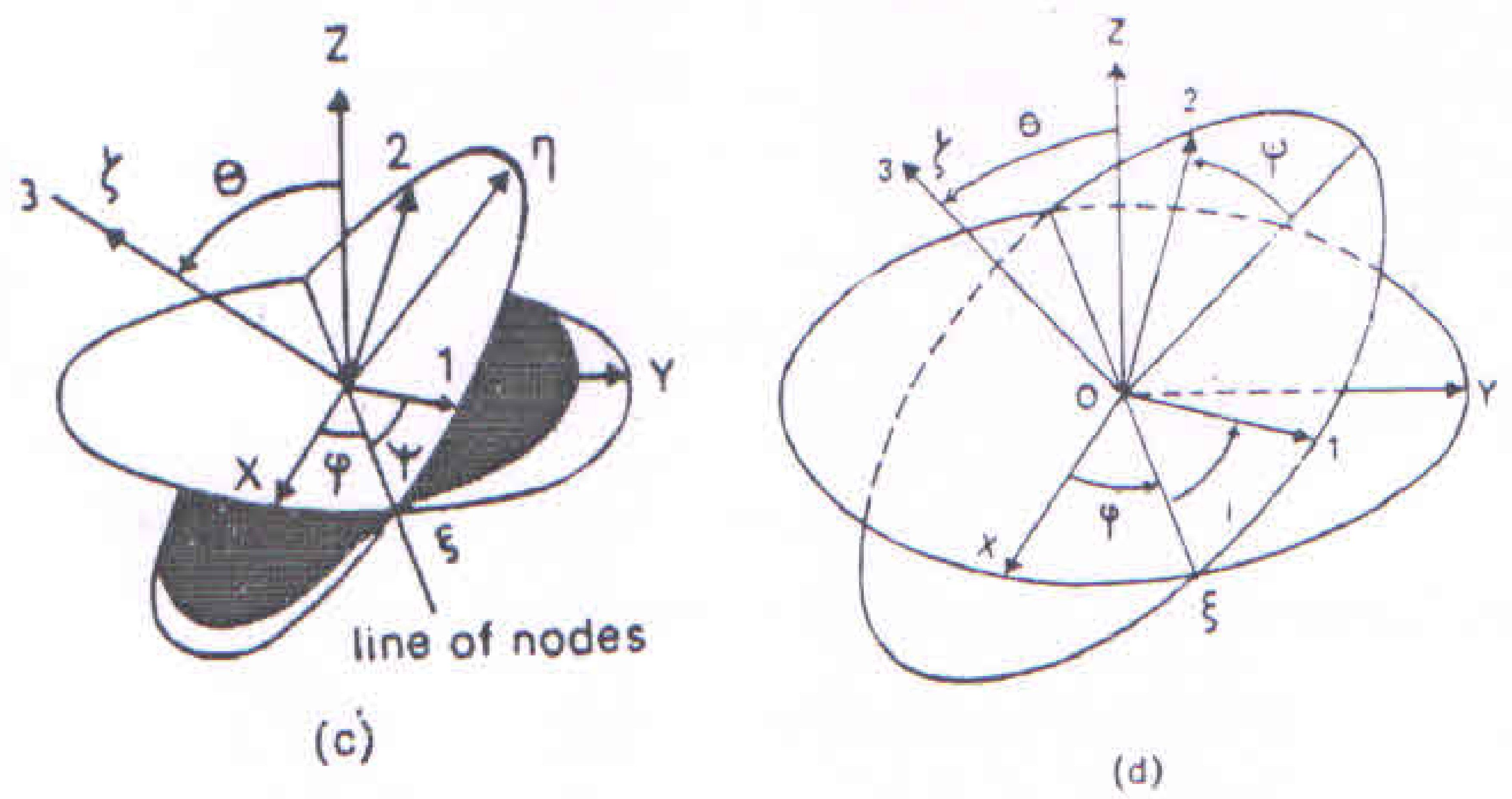
Consider the motion of a rigid body to a body frame of reference and the axes of which are coincident with the principal axes of the body. These axes rotate along with the body.

We now introduce axes fixed in space and label them $O(XYZ)$. These axes form an inertial frame of reference will be denoted by $O(123)$. The third axis will be the axis of symmetry.

The orientation of a rotating body can be completely specified by giving three angles called Euler's angles. These angles are then the generalized coordinates used for locating the rigid body.

Let the fixed axes be rotated through angle ϕ about the Z −axis as shown in Fig.(2.12a). The X −axis then takes orientation $O\xi$ called the line of nodes. Now rotate the XY −plane about the line of nodes through angle θ as shown in Fig.(2.12b).





The Y and $Z -$ axes in the new position are denoted by $O\eta$ and $O\zeta$. Thus, we get a new system of coordinates$(\xi\eta\zeta)$. Then, a rotation through angle Ψ is carried out about the ζ −axis, as shown in Fig.(2.12c) to bring the axes of coordinates to the final configuration $O(123)$ as shown in Fig.(2.12d).

In this configuration, these axes are coincident with the axes of the body frame of reference. The rotation ϕ, θ and Ψ are carried out in the same order. These are the three angles used to locate the body with reference to the fixed axes.

It should be noted that $\zeta(3)$, Z and η axes are in one plane. As the body rotates, the 1, 2, 3 axes and the ξ , η , ζ axes also rotate relative to the X, Y, Z axes.

Angular velocities $\dot{\phi}, \dot{\theta}$ and $\dot{\Psi}$ are directed along the axes of Z, ξ and 3 respectively. If the body is spinning about axis 3 , then $\dot{\Psi}$ represents the spin angular velocity.

- If the body is spinning about axis 3, then $\dot{\Psi}$ represents the spin angular velocity.
- When,$\dot{\theta} = 0$, angular velocity $\dot{\phi}$ represents the angular velocity with which axis 3 is rotating about the $Z -$ axis. Hence, $\dot{\phi}$ represents the precessional velocity.
- Axis 3 moves on the surface of a cone which has semi-vertical angle θ and OZ as its axis. Angular velocity $\dot{\theta}$ indicates that axis 3 may not remain on the surface of the cone. This motion is called nutation.
- Thus, in general, a rotating body can perform three types of motion. (i) spin motion (ii) precessional motion, and (iii) nutational motion.

Let the angular velocity of the body be $\vec{\omega}$ which has components ω_1, ω_2 and ω_3 about the principal axes of the body frame of reference. We can resolve $\vec{\omega}$ along the ξ , η , ζ axes. The axes (123)rotate with an angular velocity $\dot{\Psi}$ about axis 3 relative to the $\xi\eta\zeta$ −axes.

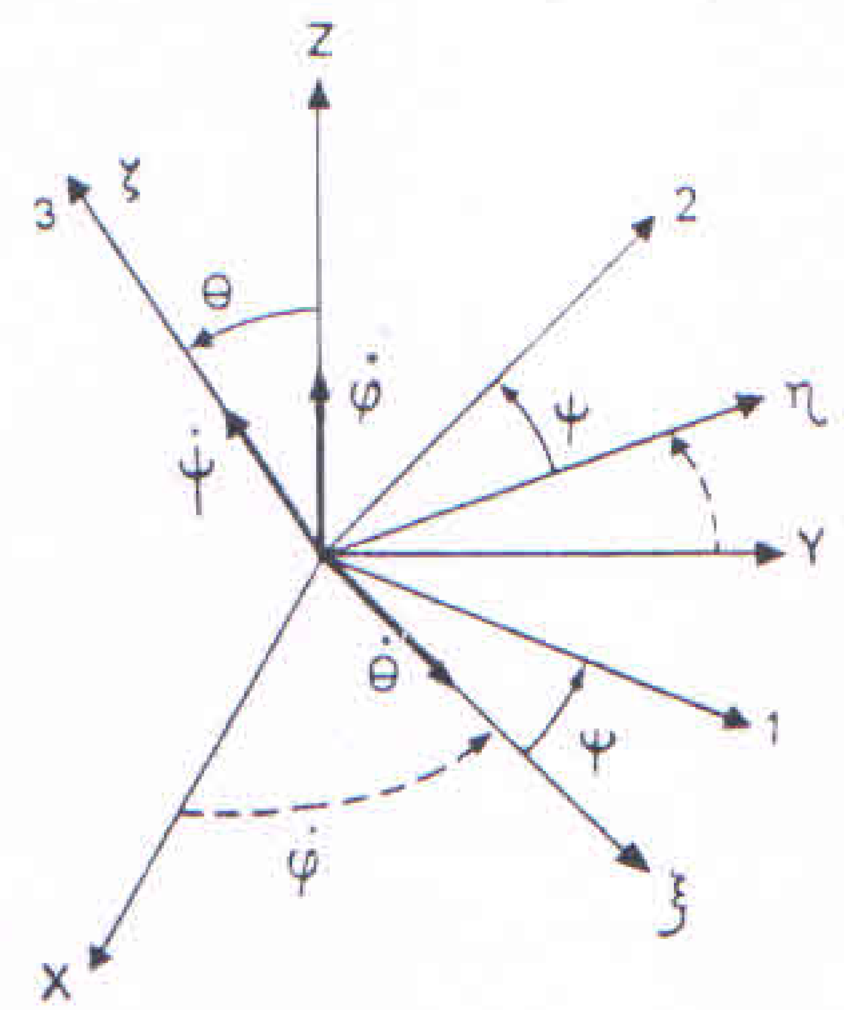


Fig.: 2.13

From above Fig. 2.13, we can write

$$\omega_\xi = \dot{\theta}\,, \qquad \omega_\eta = \dot{\phi}\, sin\theta\,, \quad \omega_\zeta = \dot{\Psi} + \dot{\phi}\, cos\theta \qquad ...(2.98)$$

These components resolved along the 1,2,3 axes and we get.

$$\left.\begin{aligned} \omega_1 &= \omega_\xi\, cos\Psi + \omega_\eta\, sin\Psi = \dot{\theta}\, cos\Psi + \dot{\phi}\, sin\theta\, sin\Psi \\ \omega_2 &= -\,\omega_\xi\, sin\Psi + \omega_\eta\, cos\Psi = -\dot{\theta}\, sin\Psi + \dot{\phi}\, sin\theta\, cos\Psi \\ \omega_3 &= \omega_\xi\; = \dot{\Psi} + \dot{\phi}\, cos\theta \end{aligned}\right\} \qquad ...(2.99)$$

The kinetic energy of the body is

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \qquad ...(2.100)$$

For a symmetric top $I_1 = I_2$ and the kinetic energy becomes

$$T = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\Psi} + \dot{\phi}\, cos\theta\right)^2 \qquad ...(2.101)$$

Equation (2.101) shows that, for a symmetric top, kinetic energy T depends upon only one generalised coordinate θ.

Motion of a Symmetrical Top:

Consider the motion of a symmetric top spinning about the axis of symmetry like axis-3. The top is fixed at its lower tip and is acted upon by gravitational torque as shown in fig:2.14. Let the centre of mass of the top at distance l from its tip.

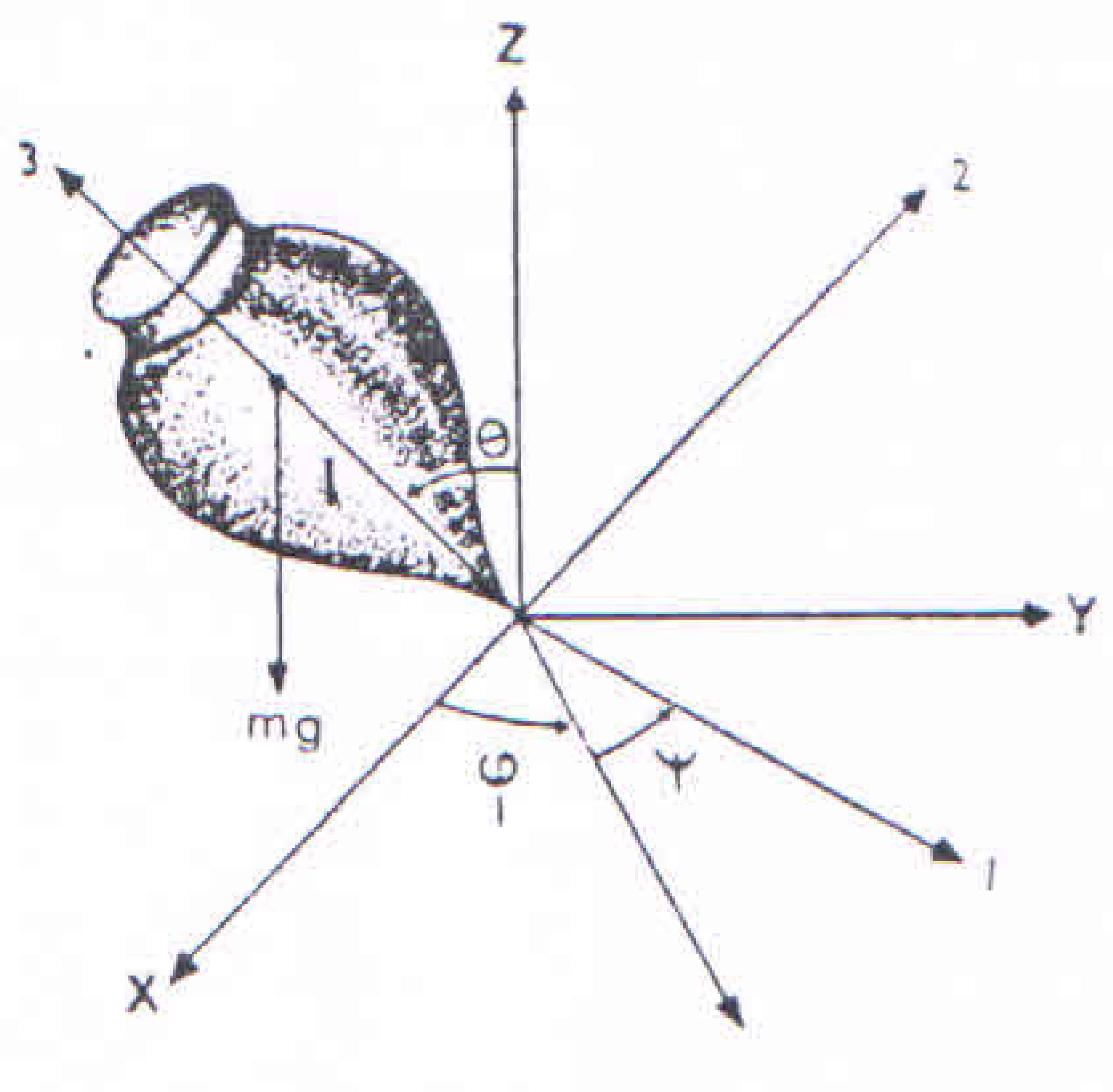


Fig.: 2.14

The Lagrangian function for the top is given by

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta\right) + \frac{1}{2}I_3\left(\dot{\Psi} + \dot{\phi}\, cos\theta\right)^2 - mgl\, cos\theta \qquad ...(2.102)$$

Here, ϕ and Ψ are absent. Hence p_ϕ and p_Ψ must be the constant of motion.

The third constant of the motion is the total energy E.

Since, $\frac{\partial L}{\partial t} = 0$, i.e. the Lagrangian does not depend upon time. Thus, the first integrals of the motion are

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\phi}\, cos\theta)$$

$$\therefore\ p_\psi = I_3\omega_3 \qquad \text{... (2.103)}$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = I_1\dot{\phi}\, sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\, cos\theta)\, cos\theta \qquad \text{... (2.104)}$$

The total energy is given by

$$E = T + V$$

$$\therefore\ E = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\, cos\theta)^2 + mgl\, cos\theta$$

$$\therefore E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\dot{\phi}^2 sin^2\theta + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\, cos\theta)^2 + mgl\, cos\theta \qquad \text{... (2.105)}$$

This expression of total energy E can be written in terms of momenta p_ϕ and p_ψ.

Using equation (2.103) in (2.104), we have

$$p_\phi = I_1\dot{\phi}\, sin^2\theta + p_\psi\, cos\theta$$

$$\therefore\ \dot{\phi} = \frac{p_\phi - p_\psi\, cos\theta}{I_1 sin^2\theta} \qquad \text{... (2.106)}$$

Using equation (2.103) in (2.105), we have

$$E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\dot{\phi}^2 sin^2\theta + \frac{{p_\psi}^2}{2I_3} + mgl\, cos\theta \qquad \text{... (2.107)}$$

Substituting the value of $\dot{\phi}$ from equation (2.106) in (2.107), we get

$$E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_1\left[\frac{p_\phi - p_\psi\, cos\theta}{I_1 sin^2\theta}\right]^2 + \frac{{p_\psi}^2}{2I_3} + mgl\, cos\theta$$

$$\therefore\ E = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\psi\, cos\theta)^2}{2I_1 sin^2\theta} + \frac{{p_\psi}^2}{2I_3} + mgl\, cos\theta \qquad \text{... (2.108)}$$

Since the term $\frac{{p_\psi}^2}{2I_3}$ on the right hand side of equation (2.108) is constant.

We introduced another constant E' denoted by

$$E' = E - \frac{{p_\psi}^2}{2I_3} = \frac{1}{2}I_1\dot{\theta}^2 + \frac{(p_\phi - p_\psi\, cos\theta)^2}{2I_1 sin^2\theta} + mgl\, cos\theta$$

$$\therefore\ E' = E - \frac{{p_\psi}^2}{2I_3} = \frac{1}{2}I_1\dot{\theta}^2 + V(\theta) \qquad \text{... (2.109)}$$

Where,

$$V(\theta) = \frac{(p_\phi - p_\psi\, cos\theta)^2}{2I_1 sin^2\theta} + mgl\, cos\theta \qquad \text{... (2.110)}$$

Here, $V(\theta)$ is the effective potential energy. It depends upon θ alone.

From equation (2.109), we have

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{2}{I_1}[E' - V(\theta)]} \qquad \text{... (2.111)}$$

The solution of above equation is,

$$t(\theta) = \int \frac{d\theta}{\sqrt{\frac{2}{I_1}[E' - V(\theta)]}} \qquad \text{... (2.112)}$$

Knowing $t(\theta)$ or $\theta(t)$, we can obtain the value of $\emptyset(t)$ and $\Psi(t)$ from equations (2.103) and (2.104).

This method of solution involves elliptic integrals and it is very difficult. For this, we use the method of energy consideration used in the central force field method.

Consider the variation of effective potential energy $V(\theta)$ with θ. The effective P.E ranges between $\theta = 0$ and $\theta = \pi$, and becomes infinity at the end values as seen from equation (2.110).

Now,

$$\frac{\partial V(\theta)}{\partial \theta} = \frac{(p_\emptyset - p_\Psi \cos\theta)(p_\Psi - p_\emptyset \cos\theta)}{I_1 \sin^3\theta} - mgl \sin\theta \qquad \ldots (2.113)$$

For $p_\emptyset \neq p_\Psi$, the quantity $\frac{\partial V(\theta)}{\partial \theta}$ is (i) positive for $\theta \to \pi$, and (ii) negative for $\theta \to 0$.

It has zero value between 0 and π. The variation of potential energy $V(\theta)$ with θ is shown in fig.2.15

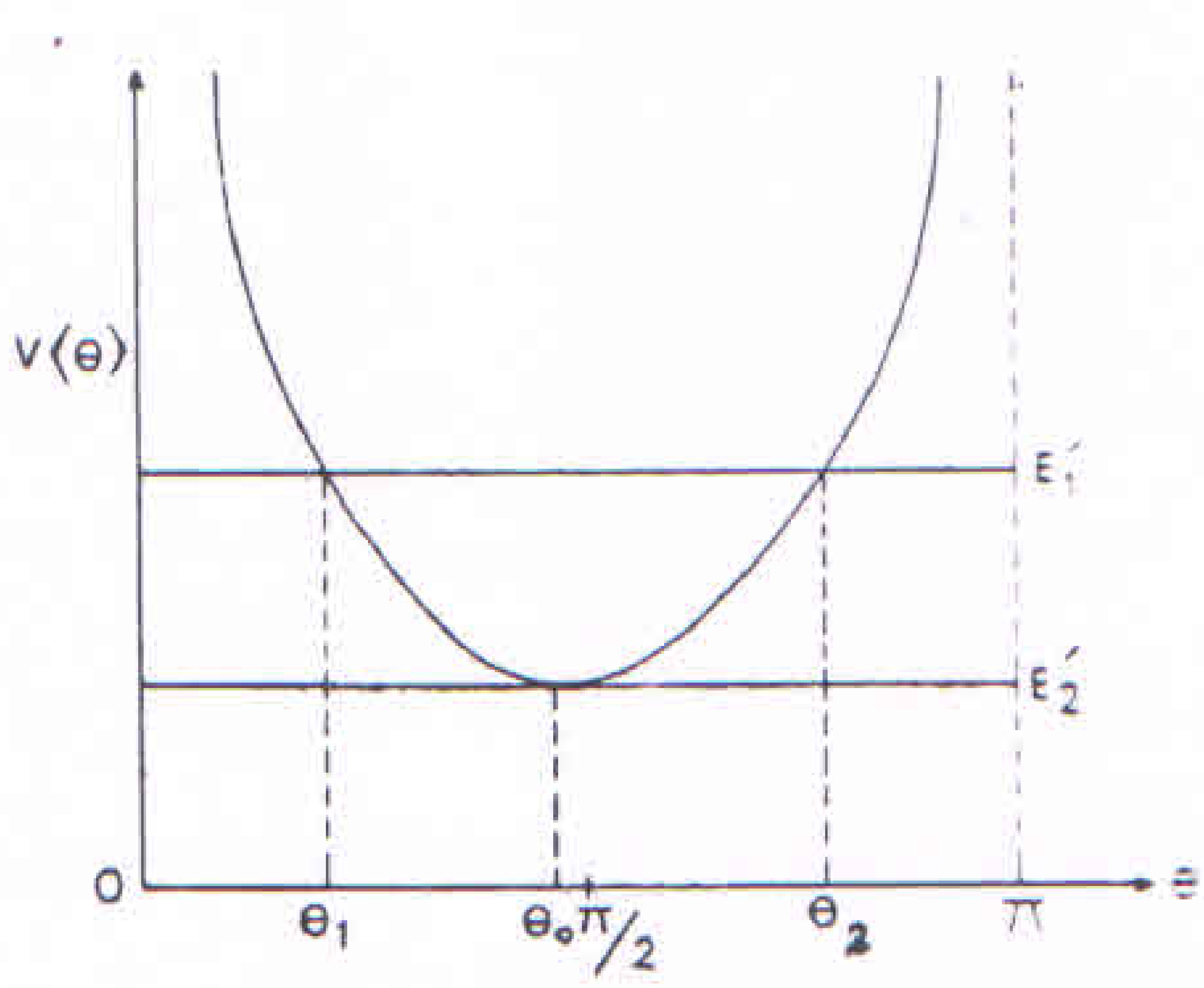


Fig.: 2.15

The value of angle θ_0, when the potential energy $V(\theta)$ is minimum, is given by equation (2.113) when $\frac{\partial V(\theta)}{\partial \theta} = 0$.

Thus,

$$(p_\emptyset - p_\Psi \cos\theta)(p_\Psi - p_\emptyset \cos\theta) = mgl I_1 \sin^4\theta \qquad \ldots (2.114)$$

If the top has energy $E' = E_1'$, then we get two values θ_1 and θ_2 of angle θ. These represent the two turning points. The motion of the top is now restricted between two values θ_1 and θ_2. Thus, the axis of symmetry performs nutational motion confined between values θ_1 and θ_2.

The two turning points coincide if the top has energy $E' = E_2'$ which corresponds to the minimum value of $V(\theta)$.

If the axis of symmetry is inclined at angle θ_0, the top precesses uniformly about the vertical $z-$axis.

The precessional velocity of this uniform motion is given by equation (2.106).

$$\therefore \ \dot{\emptyset}_0 = \frac{p_\emptyset - p_\Psi \cos\theta_0}{I_1 \sin^2\theta_0} \qquad \ldots (2.115)$$

Let, $p_\emptyset - p_\Psi \cos\theta_0 = b$, then equation (2.114) becomes

$$b^2 cos\theta_0 - bp_\psi sin^2\theta_0 + mgl l_1 sin^4\theta_0 = 0 \qquad ...(2.116)$$

Equation (2.116) is a quadratic in b and its roots are given by

$$b = p_\phi - p_\psi\, cos\theta_0 = \frac{p_\psi sin^2\theta_0}{2cos\theta_0}\left[1 \pm \sqrt{1 - \frac{4mgl l_1 cos\theta_0}{{p_\phi}^2}}\right] \qquad ...(2.117)$$

Since b is a real, the term under the square root sign is positive.

Hence,

$$p_\psi^2 \geq 4mgl l_1 cos\theta_0, \qquad for\ \theta_0 < \frac{\pi}{2} \qquad ...(2.118)$$

For $\theta_0 > \frac{\pi}{2}$, it is always positive.

Now, $p_\psi = I_3\omega_3$, Hence condition (2.118) becomes

$$(I_3\omega_3)^2 \geq 4mgl l_1 cos\theta_0$$

$$\therefore\ \omega_3 \geq \frac{2}{I_3}\sqrt{mgl l_1 cos\theta_0} \qquad ...(2.119)$$

This is the condition that must be satisfied by the value of ω_3 if there is to be a *precession without nutation.*

The equality in condition (2.119) gives the minimum spin angular velocity at which the top will be just able to perform precession without nutation. If the spin angular velocity is below this value, the top will not be able to perform uniform precessional motion.

For a given value of θ_0, there are two values of precessional velocity $\dot{\phi}$ given by equation (2.117).

Thus, for a given spin angular velocity of the top thrown at angle θ_0, the top will start performing precessional motion with two possible velocities in the same sense- one is greater than the other.

When velocity ω_3 and hence p_ψ is greater, i.e. in the case of a fast top, the term under the radial sign can be expanded to give

$$\dot{\phi}_{0\,fast} = \frac{p_\psi}{2I_1 cos\theta_0} = \frac{I_3\omega_3}{2I_1 cos\theta_0} \qquad ...(2.120)$$

Similarly, for the slow precession

$$\dot{\phi}_{0\,slow} = \frac{mgl}{p_\psi} = \frac{mgl}{I_3\omega_3} \qquad ...(2.121)$$

It is normally the slow precession that is observed in a rapidly spinning top.

Question Bank

Multiple choice questions:

(1) A frame of reference moving with a constant velocity relative to a fixed frame is called ___________ frame

(a) **inertial** (b)non inertial

(c) real (d) imaginary

(2) A frame of reference is accelerated relative to a fixed frame is called ___________ frame

(a) inertial (b) **non inertial**

(c) real (d) imaginary

(3) All the frames of reference that are rotating relative to a fixed frame of reference are the ___________ frame of reference

(a) inertial (b)**non inertial**

(c)real (d) imaginary

(4) If the moving frame of reference is accelerated the effective force acting on the particle is _________ than the actual force

(a) zero (b) equal

(c)**smaller** (d) higher

(5) Newton's laws of motion are valid in the two systems moving with a _______ relative velocity

(a) accelerated (b) double

(c) non uniform (d) **uniform**

(6) The term $\vec{\omega} \times (\vec{\omega} \times \vec{r})$ is called _________

(a) linear acceleration (b) angular acceleration

(c) **centripetal acceleration** (d) coriolis acceleration

(7) The term $2\vec{\omega} \times \left(\frac{d\vec{r}}{dt}\right)_{rot}$ is called _________

(a) linear acceleration (b) angular acceleration

(c) centripetal acceleration (d) **coriolis acceleration**

(8) In a rotational motion centripetal acceleration directed to _________ of the circle

(a) upwards (b) outwards

(c) inwards (d) **centre**

(9) In a cyclone the wind whirls in the _________ sense in the northern hemisphere

(a) upwards (b) downwards

(c) clockwise (d) **anticlockwise**

(10) In a cyclone the wind whirls in the _________ sense in the sourthern hemisphere

(a) upwards (b) downwards

(c) **clockwise** (d) anticlockwise

(11)In the rotation of a rigid body the directions of the angular velocity and the angular momentum are _________

(a) same (b) **different**

(c) perpendicular (d) parallel

(12)The moment of inertia is a tensor of rank _________

(a) one (b) **two**

(c)three (d) zero

(13) A rigid body have _________ degree of freedom

(a) one (b) two

(c)three (d) **six**

(14) If $I_1 = I_2 = I_3$, then the body is called _________

(a) **spherical top** (b) symmetrical top

(c)asymmetrical top (d) rotator

(14) If $I_1 = I_2 \neq I_3$, then the body is called _________

(a) spherical top (b) **symmetrical top**

(c)asymmetrical top (d) rotator

(15) If $I_1 \neq I_2 \neq I_3$, then the body is called _________

(a) spherical top (b) symmetrical top

(c)**asymmetrical top** (d) rotator

(16) If $I_1 = I_2$and $I_3 = 0$, then the body is called _________

(a) **spherical top** (b) symmetrical top

(c)asymmetrical top (d) rotator

(17) In a torque free motion of a rigid body, the _________ of the body is a constant vector

(a) angular velocity (b) linear velocity

(c)**angular momentum** (d) angular acceleration

(18) _________ must be applied to maintain the rotation of the system about given axis

(a) force (b) momentum

(c)velocity (d) **torque**

Short Questions:

1. Define rigid body
2. Define frame of reference
3. Define inertial and non-inertial frame of reference
4. Find the angular velocity of the earth
5. State the Euler's theorem
6. State the Chasles' theorem
7. Write the expressions of components of angular momentum
8. Derive the expression of kinetic energy of rotation of rigid body
9. Show that the directions of the angular velocity and the angular momentum are different
10. Define spherical top and asymmetric top
11. Define symmetrical top and rigid rotator
12. What you mean by torque free motion
13. Define precessional velocity

Long Questions:

1. Explain the coordinates with relative translational motion
2. Discuss the rotating coordinate systems and derive the expressions of velocity and acceleration of the particle.
3. Write note on Coriolis force
4. Explain the motion of the earth with necessary diagrams
5. Discuss the effect of Coriolis force on a freely falling particle
6. State and prove Euler's theorem
7. Derive the expressions of angular momentum and kinetic energy
8. Discuss the inertia tensor of rigid body
9. Derive the Euler's equations of the motion and find the relation between the rate at which work done by the torque and the rate of change of kinetic energy

10. Discuss the torque free motion of a rigid body and derive the expression $A^2 = \frac{L^2 - 2I_3T}{I_1(I_1 - I_3)}$
11. Discuss the Euler's angles of the rigid body with neat diagrams
12. Discuss the motion of a symmetrical top and derive the expressions of its total energy and precessional velocity